DIFFERENTIAL EQUATIONS

A differential equation is an equation involving independent variable, dependent variable and the derivatives of the dependent variable. In otherwords, if x is an independent variable, and y is dependent on x, then any equation involving x, y and $\frac{dy}{dx}$ is called a differential equation e.g.,

(i) $\frac{dy}{dx} + x\cos x = x$

(ii) $\left(\frac{dy}{dx}\right)^2 + 6y^2 = 8x$

(iii) $\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right) = -5x.$

Ordinary differential equation is a differential equation in which derivatives are involved with respect to a single independent variable.

Order of a differential equation is the order of the highest derivative appearing in the equation e.g.,

$$\frac{dy}{dx} + x\cos x = x$$

is a differential equation of first order. But the order of the differential equation (iii) as above is 2 as it involves the derivative of second order *viz.*, $\frac{d^2y}{dx^2}$.

Degree of a differential equation is the degree of the highest order derivative after the equation has been freed from the radical and fraction e.g.,

$\left(\frac{dy}{dx}\right)^2 + 6y^2 = 7x$ is 2 but degree of the diff. eq.

$\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$ is obtained after removing the radical and fractions. It is equivalent to

$\left(\frac{d^2y}{dx^2}\right)^2 = 3 + \frac{dy}{dx}$ which is 2nd order and 2nd degree.

General solution is that solution of a differential equation which contains the number of arbitrary constants equal to the order of the differential equation.

Formation of differential equations: *Rule to form the differential equation from a given equation in x and y, containig arbitrary constants.*

1. Write down the given equation.
2. Differentiate w.r.t. x successively as many times as the number of arbitrary constants.
3. Eliminate the arbitrary constants from the equations of the above two steps.

The resulting equation is the required differential equation.

Different forms of differential equations

Type I. Differential equation $\frac{dy}{dx} = f(x)$

we have $\frac{dy}{dx} = f(x)$, here we treat $\frac{dy}{dx}$ as $dy \div dx$

$\therefore\ dy = f(x)\, dx$

Integrating both sides, we get

$$\int dy = \int f(x)\, dx + c$$

where c is a constant of integration

$$\therefore\ y = \int f(x)\, dx + c$$

Example. *Solve* $\frac{dy}{dx} = \sin x - x$

$$dy = (\sin x - x)\, dx$$

Integrating both sides, we get

$$\int dy = \int (\sin x - x)\, dx$$

$$\int dy = \int \sin x\, dx - \int x\, dx + c$$

$$y = -\cos x - \frac{x^2}{2} + c$$

Type II. Variable Separable.

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable.

Rule to solve an equation in which the variables are separable.

Consider the equation $\frac{dy}{dx} = XY$, where X is a function of x only and Y is a function of y only.

1. Given differential equation is $\frac{dy}{dx} = XY$

2. $\frac{dy}{Y} = X\, dx$ i.e., variables have been separated.

3. Integrating both sides $\int \frac{dy}{Y} = \int X\, dx + c$

   where c is an arbitrary constant.

Example. *Solve* $dy + xy\, dx = x\, dx$.

Sol. On separating the variables, we have

$$\frac{dy}{1-y} = x\, dx$$

Integrating both sides, we get

$$\int \frac{dy}{1-y} = \int x\, dx + c$$

or $$-\log(1-y) = \frac{x^2}{2} + c$$

or $\frac{x^2}{2} + \log(1-y) = c.$

Type III. *Equations reducible to the form in which variables can be separated.*

Equation of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to the form in which the variables are separable.

Put $ax + by + c = z$, we have on differentiating w.r.t. x.

$$a + b\frac{dy}{dx} = \frac{dz}{dx}$$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$\therefore$ Given equation becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

or $$\frac{dz}{dx} - a = b\,f(z) \text{ or } \frac{dz}{dx} = a + b\,f(z)$$

Separating the variables $\frac{dz}{a+b\,f(z)} = dx$

which can now be integrated.

Example. *Solve* $(x-y)^2 \frac{dy}{dx} = a^2$

Sol. Put $x-y=z$, then $1-\frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = 1-\frac{dz}{dx}$

$\therefore$ Given equation becomes $z^2\left(1-\frac{dz}{dx}\right) = a^2$

or $$1-\frac{dz}{dx} = \frac{a^2}{z^2}$$

or $$\frac{dz}{dx} = 1-\frac{a^2}{z^2} = \frac{z^2-a^2}{z^2}$$

Separating the variables

$$\frac{z^2}{z^2-a^2}\,dz = dx$$

or $$\left(1+\frac{a^2}{z^2-a^2}\right)dz = dx$$

Integrating both sides,

$$z^2 + a^2 . \frac{1}{2a} \log \frac{z-a}{z+a} = x + c$$

or $$x - y + \frac{a}{2} \log \frac{x-y-a}{x-y+a} = x + c$$

or $$\frac{a}{2} \log \frac{x-y-a}{x-y+a} = y + c.$$

Type IV. *Equations of the form* $\dfrac{d^2y}{dx^2} = f(x).$

We have, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = f(x)$

Integrating w.r.t x, we get

$$\frac{dy}{dx} = \int f(x)\, dx + c_1$$

Let $\int f(x)\, dx = \phi(x)$ then $\dfrac{dy}{dx} = \phi(x) + c_1$

Again integrating w.r.t. x, we get

$$y = \int \phi(x)\, dx + c_1 x + c_2$$

Example. *Solve* $\frac{d^2y}{dx^2} = x.$

Sol. We have $\frac{d^2y}{dx^2} = x$ or $\frac{d}{dx}\left(\frac{dy}{dx}\right) = x$, Integrating

$$\int \frac{d}{dx}\left(\frac{dy}{dx}\right)dx = \int x\, dx + c_1 \text{ or } \frac{dy}{dx} = \frac{x^2}{2} + c_1$$

$$\text{or} \qquad \int \frac{dy}{dx}\cdot dx = \int\left(\frac{x^2}{2} + c_1\right)dx + c_2$$

$$\text{or} \qquad y = \frac{x^3}{6} + c_1 x + c_2.$$

First order linear equation with coefficients. A linear differential equation is that in which the dependent variables and its differential coefficients occur only in the first degree and are not multiplied together.

The standard form of linear differential equation of the first order is

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

P and Q are functions of x (and not of y) or constants.

Similarly, $$\frac{dx}{dy} + Px = Q$$

where P and Q are function of y only, is also called a linear differential equation of the first order.

Solution of differential equation $\frac{dy}{dx} + Py = Q$ is

$$ye^{\int Pdx} = \int Q . e^{\int Pdx.} . dx + c$$

or y (integrating factor *i.e.* I.F,)

$$= \int Q.(I.F.)\, dx + c$$

while evaluating the I.F. it is very useful to remember that

$$e^{\log f(x)} = f(x)$$

Illustration. *Solve* $\frac{dy}{dx} + y \tan x = \sec x.$

Sol. $P = \tan x$ and $Q = \sec x$ are functions of x only

$$\text{I.F.} = e^{\int P dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

$$y.(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$$

$$y \sec x = \int \sec x . \sec x \, dx + c$$

$$= \int \sec^2 x . \, dx + c$$

$$y \sec x = \tan x + c$$

$$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$$\therefore \quad y = \sin x + c \cos x$$