EXPONENTIAL AND LOGARITHMIC SERIES

Exponential Theorem: For all value of x, positive or negative, integral or fractional, real or complex numbers.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad \dots(i)$$

The series on the right hand side of the above relation is called the exponential series.

Some Important Results of Exponential Series e^x:

(i) When $x = 0$, the series of eqn. (i) becomes

$$e^0 = 1 + 0 + 0 + \dots = 1$$

(ii) When $x = 1$, then series of eqn. (i) becomes

$$e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

and value of e lies between 2 and 3.

i.e., $2 < e < 3$.

(*iii*) When $x = 2$, the series of eqn. (*i*) becomes

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots$$

This means that sum of the series on the right hand side is same as the square of the number e.

(*iv*) When $x = -1$, then

$$e^{-1} = \frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots.$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots.$$

The sum of this series is the reciprocal of e.

(*v*) When $x = -y$, then

$$e^{-y} = 1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \ldots$$

or $$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \quad (\because y = x)$$

(*vi*) $e^x + e^{-x} = 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right]$

(*vii*) $e^x - e^{-x} = 2\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right]$

(*viii*) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

Exponential Theorem:

$a^x = e^{x \log_e a}$

$$= 1 + \frac{x \log_e a}{1!} + \frac{x^2(\log_e a)^2}{2!} + \frac{x^3(\log_e a)^3}{3!} + \ldots$$

(for $a > 0$)

The Valve of e: The value of e lies between 2 and 3 *i.e.*, $2 < e < 3$, and hence $e = \left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$

The value of e upto ten places of decimals is found as

$$e = 2.7182818284.$$

$$\lim_{x \to \infty} (1 + 1/x)^x = \lim_{x \to \infty} (1 + x)^{1/x} = e$$

Logarithmic Series: If $-1 < x < 1$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5} + \ldots$$

This is known as Logarithmic series.

Some Important Deductions:

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

where $-1 < x \leq 1$

(ii) $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)$

where $-1 \leq x < 1$

(iii) $\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

(iv) $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right);$

where $-1 < x < 1$

(v) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ up to ∞

Properties of Logarithms: In performing, expansions are make use of the following properties of logarithms.

(*i*) $\log xy = \log x + \log y$

(*ii*) $\log \frac{x}{y} = \log x - \log y$

(*iii*) $\log x^y = y \log x$

(*iv*) $a^x = e^{x \log a}$

(*v*) $\log 1 = 0$.

The Difference between the exponential and logarithmic series:

(*i*) The exponential series is valid for all values of x. The logarithmic series is valid when $|x| < 1$, *i.e.*, $-1 < x < 1$.

(*ii*) In exponential series, the denominator of the terms involve the factorial, whereas in logarithmic series the factorial does not occur.

(*iii*) In the exponential series e^x all the terms are positive whereas in the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ ... the terms carry alternatively +ve and -ve sign.