INDEFINITE AND DEFINITE INTEGRAL

Integration is the inverse operation of differentiation: In differential calculus, we are given a function and we are required to differentiate it. In integral calculus we are required to find a function whose differential coefficient (or derivative) is given.

If the differential coefficient of a function $f(x)$ w.r.t. x is $F(x)$ then we say that the integral (or primitive) of $F(x)$ w.r.t. x is $f(x)$.

Symbolically. If $\frac{d}{dx}[f(x)] = F(x)$ then

$\int F(x)\,dx = f(x)$ and is read as "integral of $F(x)$ w.r.t. x is $f(x)$".

The symbol $\int$ (elongated S) is called the sign of integration, $f(x)$ is called the integrand and the process of finding $f(x)$ is called integration.

Constant of integration

If $\frac{d}{dx}[f(x)] = \mathrm{F}(x)$ then, $\int \mathrm{F}(x)\, dx = f(x) + c$ where c is an arbitrary constant.

The constant c is called the constant of integration. The constant of integration is generally omitted.

Note : Different methods of integrating a function may give different answers apparently, but any two such answers differ only by a constant.

Standard Forms

(*i*) $\int x^n dx = \frac{x^{n+1}}{n+1} [n \neq -1]$

i.e., increase the index of x by 1 and divide by the new index.

(*ii*) $\int \frac{1}{x} dx = \log x$

Note. In (*i*) $n = -1$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \log x$$

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a}; \int e^{x} dx = e^{x}$

(iv) $\int a^{mx} dx = \frac{a^{mx}}{m}; \int a^{x} dx = \frac{a^{x}}{\log a}$

(v) $\int \sin x \ dx = -\cos x$

(vi) $\int \cos x \ dx = \sin x$

(vii) $\int \sec^{2} x \ dx = \tan x$

(viii) $\int \operatorname{cosec}^{2} x \ dx = -\cot x$

(ix) $\int \sec x \tan x \ dx = \sec x$

(x) $\int \operatorname{cosec} x . \cot x = -\operatorname{cosec} x$

(xi) $\int 1 \ dx = \int x^{0} dx = \frac{x^{0+1}}{0+1} = x$

Important extension of elementary forms:

(*i*) All the results of the above list hold good when x is replaced by $x + a$ [a being a constant] in any formula.

$$e.g., \int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} \qquad [n \neq -1]$$

$$\int \sec(x+6)\tan(x+6)dx = \sec(x+6)$$

(ii) If x be replaced by $ax + b$ [a and b being constants] on both sides of any standard result of the above table of integrals, the standard form remains true, provided the result on R.H,S. is divided by 'a' the coefficient of x

$$i.e. \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)\,a} \qquad [n \neq -1]$$

$$= \int \text{cosec}^2(8x+3)dx = -\frac{\cot(8x+3)}{8}$$

Theorems on integration

Theorem 1. The processes of differentiation and integration neutralises each other *i.e.* $\frac{d}{dx}\left[\int f(x)dx = f(x)\right]$

Theorem 2. The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function.

i.e. $\int c\, f(x)\, dx = c\int f(x)dx$

e.g. $\int 5x^3 dx = 5\int x^3 dx = \frac{5x^{3+1}}{3+1} = \frac{5}{4}x^4$

Theorem 3. If u, v, w ... (finite number) are functions of x, then

$$f(u+v+w+\ldots)dx = \int udx + \int vdx + \int wdx$$

e.g.,

$$\int\left(a+\frac{b}{x}+\frac{c}{x^2}\right)dx = a\int 1.dx + b\int \frac{1}{x}dx + c\int \frac{1}{x^2}dx$$

$$= ax + b\log x + c\frac{x^{-2+1}}{-2+1} = ax + b\log x - \frac{c}{x}$$

Note: If the degree of the numerator of the integrand is equal to or greater than that of denominator, divide the numerator by the denominator until the degree of the remainder is less than that of denominator *e.g.*

$$\int \frac{x^3}{x+1}\,dx = \int \left(x^2 - x + 1 - \frac{1}{x+1} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log\,(x+1)$$

Difinite integral. If $\phi\,(x)$ be an integral of $f(x)$ then the quantity $\phi(b) - \phi(a)$ is called the definite integral of $f(x)$ between the limits a and b is written as $\int_a^b f(x)dx$. *It is read as integral from a to b* of $f(x)\,dx$. 'a' is called the lower limit and b the upper limit, $\phi(b) - \phi(a)$ is written as $[\phi\,(x)]_a^b$

Thus $\int_a^b f(x)dx = \left[\phi(x)\right]_a^b = \phi(b) - \phi(a)$.

Rule to evaluate $\int_a^b f(x)\,dx$

1. Integrate $\int_a^b f(x)\,dx$...

2. In the result first put x = upper limit (b), and then x = lower limit (a).
3. Subtract the second result from the first.

e.g. $$\int_3^4 \frac{dx}{x} = [\log x]_3^4 = \log 4 - \log 3$$

$$\therefore \qquad = \log \frac{4}{3} \quad \left[\because \log m - \log n = \log \frac{m}{n}\right]$$

Integration by substitution. Integration of many functions become simple by substitution of a new variable. In other words, many integrals can be evaluated in a simple way by changing the variable of the given integrand, say from the given variable x to the new variable z, the two variable x, z being connected by some relation. This process of integration is called *integration by substitution*.

For example, Let $\mathrm{I} = \int \sin^3 x \cos x \, dx$

Put $\sin x = t$, then $\cos x = \frac{dt}{dx}$

[Differentiation w.r.t. x]

or, $\cos x \, dx = dt$

$$\therefore\ \mathrm{I} = \int t^3 . \, dt = \frac{t^4}{4} + c$$

$$= \frac{1}{4} \sin^4 x + c$$

Two important forms of integrals

Theorem 1. $\int \frac{f'(x)\,dx}{f(x)} = \log [f(x)]$

Aid to memory. The integral of a fraction whose numerator is differential co-efficient of the denominator is log [denom.]

e.g., $$\int \frac{2x+3}{x^2+3x+7}\,dx$$

$$\because\ \frac{d}{dx}\left(x^2+3x+7\right) = 2x+3$$

$$\therefore\ \int \frac{2x+3}{x^2+3x+7}\,dx = \log\left(x^2+3x+7\right)$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

Theorem 2. $\int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}$

Provided $n \neq -1$

Thus if the integrand is the product of a power of a function $f(x)$ and its derivative $f'(x)$ then the integral is obtained by increasing the index of $f(x)$ by 1 and dividing the result by the new index.

$$e.g., \int \left(ax^2 + bx + c\right)^5 (2ax + b)\,dx$$

$$[\text{Form} \int [f(x)^n f'(x) dx]$$

$$= \frac{\left(ax^2 + bx + c\right)^{5+1}}{5+1}$$

$$= \begin{bmatrix} \text{Here } f(x) = \left(ax^2 + bx + c\right) \\ f'(x) = 2ax + b \end{bmatrix}$$

Remember. $\int \tan x\,dx = -\log \cos x = \log \sec x.$

Remember. $\int \cot x\,dx = \log \sin x.$

Remember. $\int \operatorname{cosec} x \, dx = \log \tan \frac{x}{2}.$

Remember. $\int \operatorname{cosec} x \, dx = \log (\sec x - \cot x)$

Remember. $\int \sec x \, dx = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right).$

Remember. $\int \sec x \, dx = \log (\sec x + \tan x).$

Five Standard Forms

(*i*) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$

(*ii*) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$

(*iii*) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}.$

(*iv*) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a}$

(*v*) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \cosh^{-1} \frac{x}{a}$

Remember. To integrate a fraction whose numerator is 1 and denominator is a homogeneous function of the second degree in $\cos x$ and $\sin x$.

1. Divide the numerator and denominator by $\cos^2 x$.
2. Put $\tan x = z$.

e.g., $\int \frac{d\theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ Dividing num. and denom. by $\cos^2\theta$

$$\int \frac{\sec^2 \theta \, d\theta}{a^2 \tan^2 \theta + b^2} = \frac{1}{a^2} \int \frac{dz}{z^2 + \frac{b^2}{a^2}}$$

[**Note.** To make the coeff. of z^2 unity]

$$= \frac{1}{a^2} \int \frac{dz}{z^2 + \left(\frac{b}{a}\right)^2} \left[\text{From} \int \frac{dx}{x^2 + a^2} \text{ Here } a = \frac{b}{a}\right]$$

$$= \frac{1}{a^2} \cdot \frac{1}{\frac{b}{a}} \tan^{-1}\left(\frac{z}{\frac{b}{a}}\right) = \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} z\right)$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan\theta\right) [\because z = \tan\theta]$$

Integration by parts

If u and v be two function of x, then

$$\int uv\, dx = u \int v\, dx - \int \frac{du}{dx}\left(\int v\, dx\right) dx$$

In words. Integral of the product of two functions = Ist function × integral of 2nd-integral of [Diff. coeff. of 1st × Integral of 2nd].

Rule to choose the factor of differentiation or the first function. If one factor of the product is a power of x take it as the first function provided the integral of the second function is handy. If however the integral of the second function is not readily available [in case of inverse circular function or inverse hyperbolic function or a logarithmic function] in that case, take that factor as the first function.

If the integrand is a single inverse circular function (or hyperbolic function) or a single logarithm, take that function as the function and unity (1) as the second function.

$$\int e^x[f(x)+f'(x)]dx = e^x f(x)$$

$$\int e^x(\sin x+\cos x)\,dx\left[\text{Here } f(x)=\sin x, f'(x)=\cos x\right]$$

$$= e^x \sin x$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int \sqrt{a^2+x^2}\;dx = \frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

$$\int \sqrt{x^2-a^2}\;dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

i.e., for

$$\int \sqrt{a^2-x^2}\;dx,\ \int \sqrt{a^2+x^2}\;dx,\ \int \sqrt{x^2-a^2}\;dx$$

$$\text{Integral} = \frac{x\times\text{Integrand}}{2}.\frac{+a^2\text{or}-a^2\text{as integrand}}{2}$$

$$\times \text{ integral of [reciprocal of integrand]}$$

Two standard integrals

$$\left.\begin{aligned}\int e^{ax}\sin bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\\ \int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\end{aligned}\right]\text{First form}$$

$$\left.\begin{aligned}\int e^{ax}\sin bx\,dx &= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right)\\ \int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right)\end{aligned}\right]\text{2nd form}$$

Important Note. $e^{\log f(x)} = f(x)$

INTEGRATION OF RATIONAL FUNCTIONS

Two standard forms

(i) $\int \frac{dx}{x^2-a^2}\left[x^2 > a^2\right] = \frac{1}{2a}\log\frac{x-a}{x+a}$

(ii) $\frac{dx}{a^2-x^2}\left[x^2 < a^2\right] = \frac{1}{2a}\log\frac{a+x}{a-x}$

$\int \frac{dx}{ax^2+bx+c}$, a be + ve then.

Case I. When $b^2 > 4ac$ then

$$= \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}$$

Case II. When $b^2 < 4ac$ then

$$= \frac{2}{\sqrt{4ac - b^2}} \tan^{-1}\left[\frac{2ax + b}{\sqrt{4ac - b^2}}\right]$$

To integrate $\int \frac{dx}{\text{Quadratic}}$

1. Make the coefficient of x^2 unity by taking the numerical coefficient of x^2 outside.

2. Complete the square in terms containing x by adding and subtracting the square of half the coefficient of x.

3. Use the proper standard form.

Note: If in a numerical problem, the discriminant of quadratic in denominator $[b^2 - 4ac]$ is +ve and a perfect square, factorise the denominator and resolve it into partial fractions then integrate, *e.g.*,

$$\int \frac{dx}{2x^2 - 2x + 1} = \frac{1}{2}\int \frac{dx}{x^2 - x + \frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}} = \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x - \frac{1}{2}}{\frac{1}{2}} = \tan^{-1}(2x - 1).$$

Method to integrate $\int \frac{\text{Linear}}{\text{Quadratic}}\,dx.$

1. Put linear $= \lambda \frac{d}{dx}(\text{quadratic}) + \mu$
2. Equate the coefficients of x and constant terms on both sides to find λ and μ.

The above two steps are taken to break the given fraction into two fractions such that in one the numerator is the derivative of denominator and in the other, numerator is a constant.

e.g., $\int \frac{2x}{x^2 + 2x + 2}\,dx$

Let $2x = \lambda \frac{d}{dx}\left(x^2 + 2x + 2\right) + \mu$

i.e. $2x = \lambda\,(2x + 2) + \mu$

Equating coeffi. x, $2 = 2\lambda \quad \therefore \quad \lambda = 1$

Equating constant term, $0 = 2\lambda + \mu$

$\mu = -2\lambda = -2$

$$\therefore \int \frac{2x}{x^2 + 2x + 2}\,dx = \int \frac{\lambda(2x + 2) + \mu}{x^2 + 2x + 2}\,dx$$

$$= \lambda \int \frac{2x + 2}{x^2 + 2x + 2}\,dx + \mu \int \frac{dx}{x^2 + 2x + 2}$$

$$= \lambda \log\left(x^2 + 2x + 2\right) + \mu \int \frac{dx}{(x + 1)^2 + 1}$$

$$= \log\left(x^2 + 2x + 2\right) - 2\tan^{-1}(x + 1)$$

$[\because \lambda = 1, \mu = -2]$

Integration of irrational functions

Rule to integrate

$$\int \frac{dx}{\sqrt{\text{Quadratic}}} \text{ or} \int \sqrt{\text{Quadratic}}\;dx$$

1. Make the coeffi. of x^2 unity by taking its numerical coeffi. outside the square root sign.
2. Complete the square in terms containing x by adding and subtracting the square of half the coeffi. of x.
3. Use proper standard form.

Working rule to calculate the area under a plane curve.

Step I. Make a rough sketch of the graph of the given function.

Step II. Make the region whose area is to be calculated.

Step III. Set up the definite integral for the area in such a way so that limits of integration are so chosen that the independent variable varies throughout the region.

Step IV. Evaluate the definite integral set up in step III.

Reduction formula: A formula which connects an integral which cannot otherwise be evaluated, with and another integral of the same type but of lower degree is called a reduction formula.

In this method, we go on reducing the power till we get a power whose intgral is knwon or which can be integrated easily. Reduction formula is generally obtained by the method of Integration by parts.

(*i*) Reduction formula for $\int n^x e^{ax}\, dx$ is

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$$

and $I_{n-1} = \frac{1}{a} x^{n-1} e^{ax} - \frac{n-1}{a} I_{n-2}$

[Replacing n by $n-1$]

In this way we get I_{n-2}, I_{n-3}

(*ii*) Reduction formula for $\int \sin^n x\, dx$ is

$$I_n = \frac{-\sin^{n-1} x . \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

(*iii*) Reduction formula for $\int \cos^n x\, dx$ is

$$I_n = \frac{\cos^{n-1} x . \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

(*iv*) Reduction formula for $\int \tan^n x\, dx$ is

$$I_n = \frac{\cos^{n-1} x}{n-1} - I_{n-2}$$

(v) Reduction formula for $\int \cot^n x \, dx$ is

$$I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

(vi) Reduction formula for $\int \sec^n x \, dx$ is

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

(vii) Reduction formula for $\int \operatorname{cosec}^n x \, dx$ is

$$I_n = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

(viii) Reduction formula for $\int x^n \sin x \, dx$ is

$$I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}$$

(ix) Reduction formula for $\int x^n \cos x \, dx$ is

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$$

(x) Reduction formula for

$\int \sin^m x \cos^n x \, dx$ **is**

$$I_{m,n} = \frac{\cos^{n-1} x . \sin^{m+1} x}{m+n} + \frac{n-1}{m+1} I_{m,n-2}$$

Definite Integral

First Fundamental Theorem of Integral Calculus: Let $f(x)$ be a continuous function of x for $a \le x \le b$ and

$A(x) = \int_o^x f(x)\, dx$, then $A'(x) = f(x)$ for all x in $[a, b]$ and $A(a) = 0$

Second Fundamental Theorem of Integral Calculus: Let f be a continuous function defined on the interval $a \le x \le b$ and ϕ be an anti-derivative of f, then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

In words the above theorem tells that

$\int_a^b f(x)\, dx$ = (value of an antiderivative at b, the upper limit) – (value of the same anti-derivative at a, the lower limit)

Note: We often write $\phi(b)-\phi(a)$ as $\left[\phi\,(x)\right]_a^b$

Definite Integral as a limit of a Sum: If $f(x)$ be a single valued continuous function defined in the interval (a, b) where $a < b$ and the interval (a, b) is divided into n equal parts of each length h so that $nh = b-a$ then we define

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\Big[f(a)+f(a+h)+f(a+2h)+\ldots + f\big(a+\overline{n-1}h\big)\Big]$$

OR

$$\int_a^b f(x)\,dx = (b-a)\lim_{h\to\infty}\frac{1}{n}\Big[f(a)+f(a+h)+f(a+2h)+\ldots + f\big(a+\overline{n-1}\,h\big)\Big]$$

where $h = \dfrac{b-a}{n}$

is called the definite integral of $f(x)$ between the limits $x = a$ and $x = b$.

Some Properties of Definite Integrals

(*i*) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if $a < c < b$

(iv) $\int_0^a f(x)\,dx = \int_o^a f(a-x)\,dx$

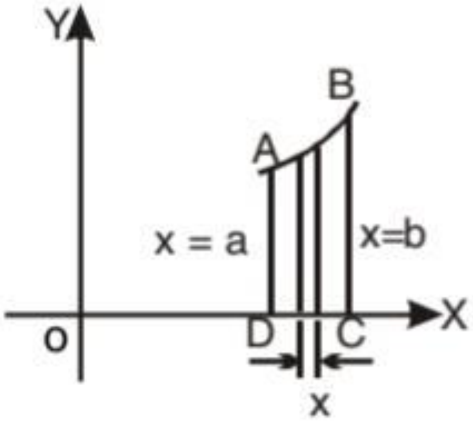
(v) $\int_{-a}^a f(x)\,dx = 0$, when $f(x)$ is an odd function

$= 2\int_0^a f(x)\,dx$, when $f(x)$ is an even function

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x) = 0$

if $f(2a - x) = -f(x)$

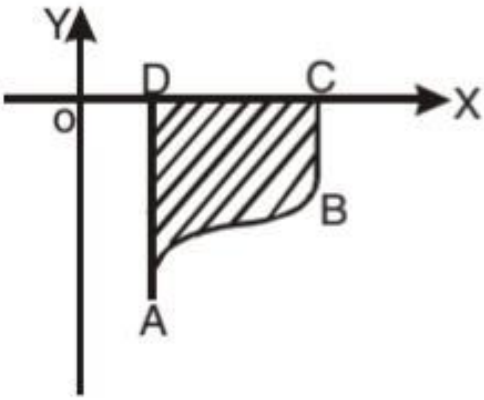
Definite Integral as Area under the curve: Let $f(x)$ be finite and continuous in $a \le x \le b$. Then area of the region bounded by x–axis, $y = f(x)$ and the ordinates at $x = a$ and $x = b$ (i.e., Area ABCD) is equal to



$$\int_a^b f(x)\,dx$$

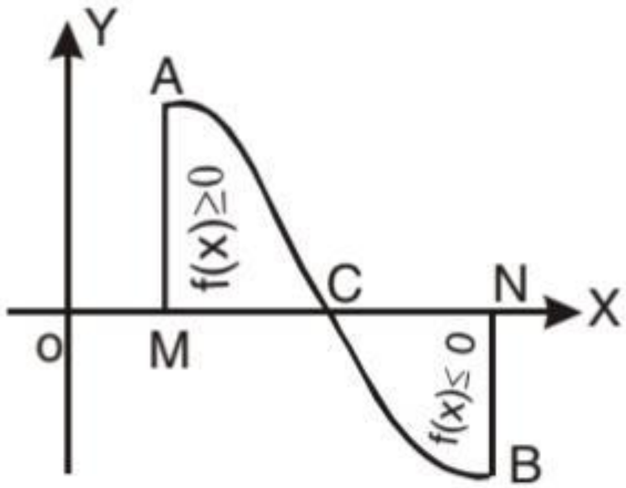
Remark : In the above fig. we assumed that $f(x) \geq 0$ for all x in $a \leq x \leq b$. However if

(*i*) $f(x) \leq 0$ for all x in $a \leq x \leq b$ then area bounded by x–axis, the curve $y = f(x)$ and the ordinate $x = a$ to $x = b$ is given by



$$= -\int_a^b f(x)\,dx$$

(*ii*) If $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $b \leq x \leq c$, then area bounded by $y = f(x)$, x–axis and the ordinates $x = a$, $x = b$ is

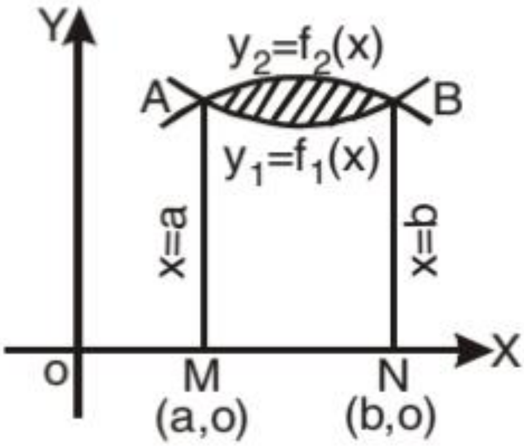


$$= \int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

(*iii*) The area of the region bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the ordinates $x = a$ and $x = b$ is given by

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$



where $f_2(x)$ is y of upper curve and $f_1(x)$ is y of lower curve *i.e.*, Required area

$$= \int_a^b \left[f_2(x) - f_1(x)\right] dx$$