LINEAR PROGRAMMING

Linear Inequations in Two Variables: A statement of any one of the following types :

(i) $ax + by + c > 0$

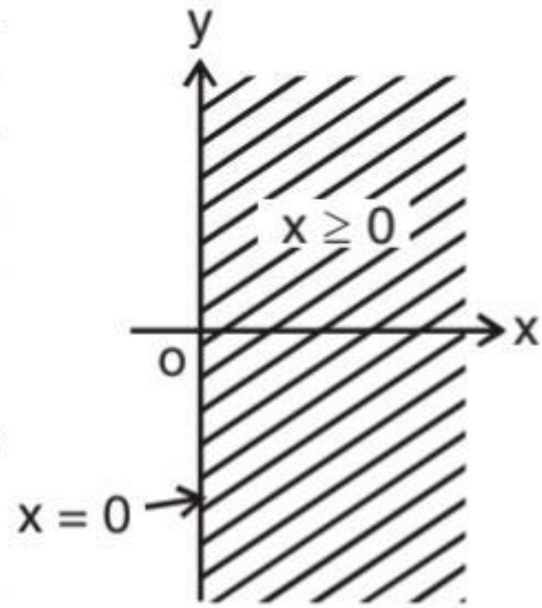
(ii) $ax + by + c \geq 0$

(iii) $ax + by + c < 0$

(iv) $ax + by + c \leq 0$

where $a, b, c \in R$ and $a^2 + b^2 \neq 0$, is called a linear inequation (for inequality) in two variables x and y.

An ordered pair (a, b) of real numbers may or may not satisfy a given inequation. The set of all ordered pairs, which satisfy a given inequation, is called the solution set of the given inequation. Also, we know that there is one-one correspondence between the ordered pairs of real numbers and the points of the co-ordinated plane, therefore, it is possible to represent the solution set of a given inequation (in two variables) by the points of a co-ordinated plane. The set of all points whose co-ordinates satisfy a given inequation is called the graph of the inequation.



Let us consider the inequaliy $x \geq 0$. The set of this inequality is $\{(x, 0) : x \in R, x \geq 0\}$ and hence the graph of $x \geq 0$ is the set of all points of the XOY-plane whose abscissae (*i.e.*, x-axis) are non-negative *i.e.*, the set of all points which lie one Y-axis or on the right hand side of Y-axis. The graph of $x \geq 0$ is shown shaded in fig.

In general, to find the graph of an inequation, we note that $ax + by + c = 0$ represents a straight line, which divides the XOY-plane into two halves, one half is the graph of $ax + by + c > 0$ and the other of $ax + by + c < 0$. The illustrate the method of finding the graph of an inequation, we consider the following illustration.

Illustration: Draw the graph of the inequation $2x + 3y \geq 6$

Solution: The give inequality is

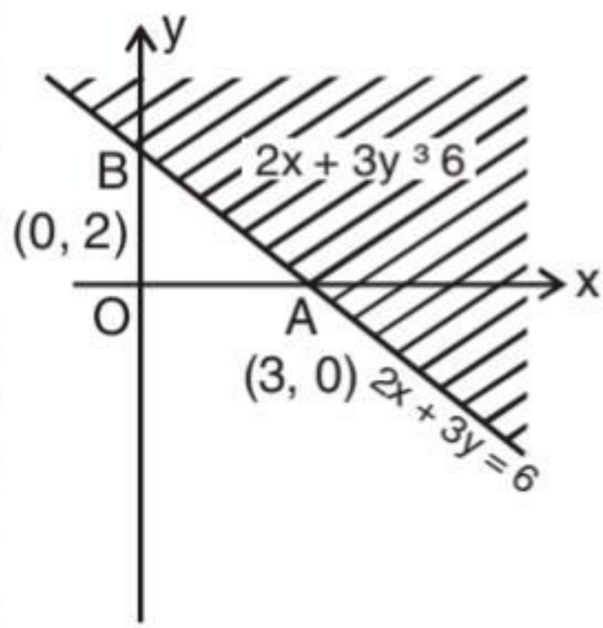
$$2x + 3y \geq 6 \qquad ...(1)$$

Consider the equation $2x + 3y = 6$. To find the straight line represented by $2x + 3y = 6$, we observe that

A (3, 0) and B (0, 2) lie on the line (Putting $x = 0$, we get $2 \times 0 + 3y = 6 \Rightarrow y = 2$ and on putting $y = 0$, we get $2x + 3 \times 0 = 6 \Rightarrow x = 3$).

Hence, the graph of $2x + 3y = 6$ is the set of all points on the line containing the points A and B.

Further we notice that O (0, 0), which lies below the line AB does not satisfy $2x + 3y \geq 6$, [$\because 2 \times 0 + 3 \times 0 = 0 < 6$], therefore, the graph of $2x + 3y \geq 6$ is that half of XOY-plane which lies above the line AB (Including the points of the line AB). The required graph is shown shaded in fig.



Thus, solving the inequation $ax + by \leq c$ by graphical method involves the following steps:

Step I. Consider the equation $ax + \text{by} = c$ and plot the resulting line. In case of strict inequalities < or >, draw the line dotted, otherwise mark it thick.

Step II. Choose a point [If possible (0, 0)], not lying on this line. Substitute its coordinates in the inequation. If the inequation is satisfied, then shade the portion of the plane which contains the chosen points; otherwise shade the portion which does not contain this point.

The shaded portion represents the solution set. The dotted line is not a part of shaded region while thick line is a part of it.

Simultaneous Inequations: The solution set of a system of linear inequations in two variables is the set of all points (x, y) which satisfy all the inequations in the system simultaneously. So, we find the region of the plane, common to all the portions comprising the solution sets of given inequations. When there is no region common to all the solution of the given inequations, we say that the solution set is empty.

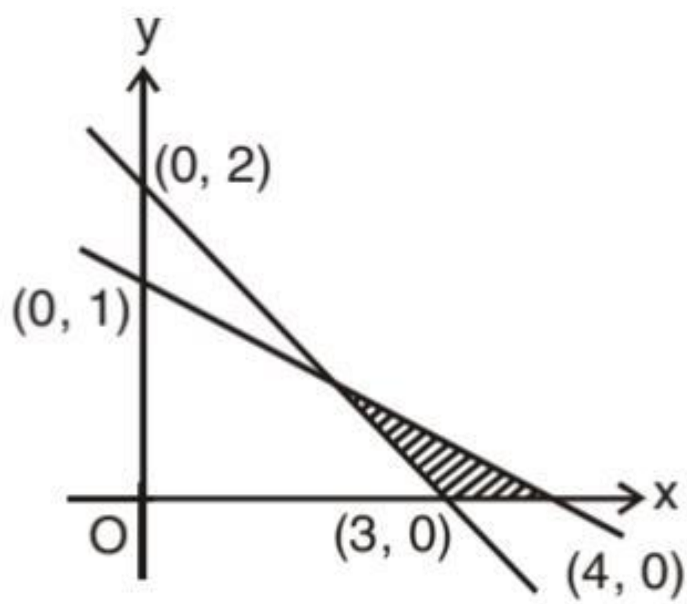
The linear inequations are also known as linear constraints.

Illustration: Draw the diagram of the solution set of the inequations $2x + 3y \geq 6$, $x + 4y \leq 4$, $x \geq 0$ and $y \geq 0$.

Solution: Consider the equations,

$2x + 3y = 6$, $x + 4y = 4$, $x = 0$ and $y = 0$.

Now $2x + 3y = 6 \Rightarrow \frac{x}{3} + \frac{y}{2} = 1.$



This meets the axes at (3, 0) and (0, 2). Join these points and draw a thick line. Clearly, the portion not containing (0, 0) represents the solution set of the inequation, $2x + 3y \geq 6$.

Agian $x + 4y = 4 \Rightarrow \frac{x}{4} + \frac{y}{1} = 1.$

This line meets the axes at (4, 0) and (0, 1). Join these points and draw a thick line. Clearly, the portion containing (0, 0) represents the solution set of the inequation $x + 4y \leq 4$

Clearly, $x \geq 0$ is represented by y-axis and the portion on its right hand side.

Also $y \geq 0$ is represented by x-axis and the portion above x-axis.

Hence, the shaded region represents the solution set of the given inequation.

Linear Programming: The idea of linear programming was first introduced by a Russian mathematician L.V. Kantorvich. In 1947, George B. Dantzig developed a superior technique of computation popularly known as the "simplex method". He developed it for the purpose of scheduling highly complex procurement activities of the United Sates Air Force. The development of electronic computers has made a significant role in the growth of linear programming because computers can easily and quickly solve complicated problems which may not be solved otherwise.

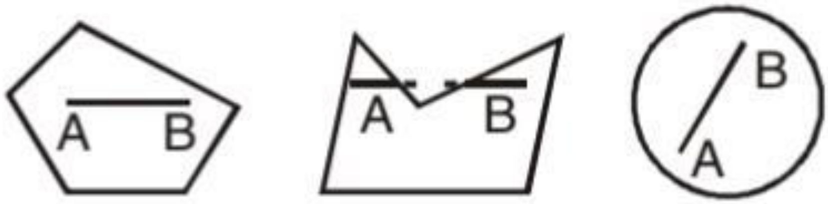
Meaning: Programming means systematic planning or decision making. It is a technique for solving optimization (maximization or minimization) problems subjects to certain constrain. Out of all permissible allocations of resources, we have to decide one which will minimize the total cost or maximize the total profits. Linear Programming is a device which is used in decision making in business for obtaining optimum values of quantities subject to certain constraints when the relationship involved in the problem are linear.

Methods of Solving linear Programming Problems: There are two methods of solving a linear programming problem.

1. Graphical Method and
2. Simplex method

The simplex method is beyond the scope of this book. We shall explain the graphical method of solving a linear programming problem. For this we need the following back ground.

(*i*) A set S of points in a plane is said to be convex, if the line segment joining any two points in it, lies in it completely, *i.e.*, if we take any two points A and B in the set S, then the segment $[AB]$ lies in S. A circle, a square, a rhombus, a polygon are examples of convex set. In fig. (*i*) and (*iii*) are convex sets, where as (*ii*) is not a convex set.



(i) (ii) (iii)

(*ii*) If we have a system of linear inequation in two variables, then the set of points (x, y) for which all the inequations of the system hold true is either empty, or convex region bounded by straight lines (a Convex Polygon) or an unbounded region with straight line boundaries.

(*iii*) The set of points, whose co-ordinates satisfy the constraints of a linear programming problem, is said to be the feasible region.

(*iv*) The optimum value (maximum or minimum) of the objective function is obtained at a vertex of the feasible region. If there are more than one point (vertices) where the objective function is optimum (max, or min.), then every point on the line segment joining any two such vertices optimizes the objective function.

Thus, solving a linear programming problem by the graphical method involves the following steps.

First step. Plot the graph of the inequalities describing the various constraints (structural) on the graph paper.

Second step. Find the portion of the graph paper in the first quadrant ($\because$ of non-negativity constraints) which is common to the graphs plotted in the first step. Locate the extreme points (*i.e.*, corner points) of this region (known as feasible region).

Third step. Find the value of the objective function corresponding to each corner point. The points which corresponds to the optimum (*i.e.*, max. or min.) value of the objective function is (are) required solution (s) of the given linear programming problems.

Remark. Though (0, 0) may be a corner point of the feasible region in some problem, but it is not to be examined for the optimum solution.

Illustration: Find the maximum and minimum values of $5x + 2y$ subject to constraints

$-2x - 3y \leq -6$ *i.e.*, $2x + 3y \geq 6$

$x - 2y \leq 2$

$6x + 4y \leq 24$

$-3x + 2y \leq 3$

$x \geq 0$ and $y \geq 0$

Solution: The bounding lines for feasible region are

$$2x+3y=6 \Rightarrow \frac{x}{3}+\frac{y}{2}=1 \qquad ...(1)$$

$$x-2y=2 \Rightarrow \frac{x}{2}+\frac{y}{-1}=1 \qquad ...(2)$$

$$6x+4y=24 \Rightarrow \frac{x}{4}+\frac{y}{6}=1 \qquad ...(3)$$

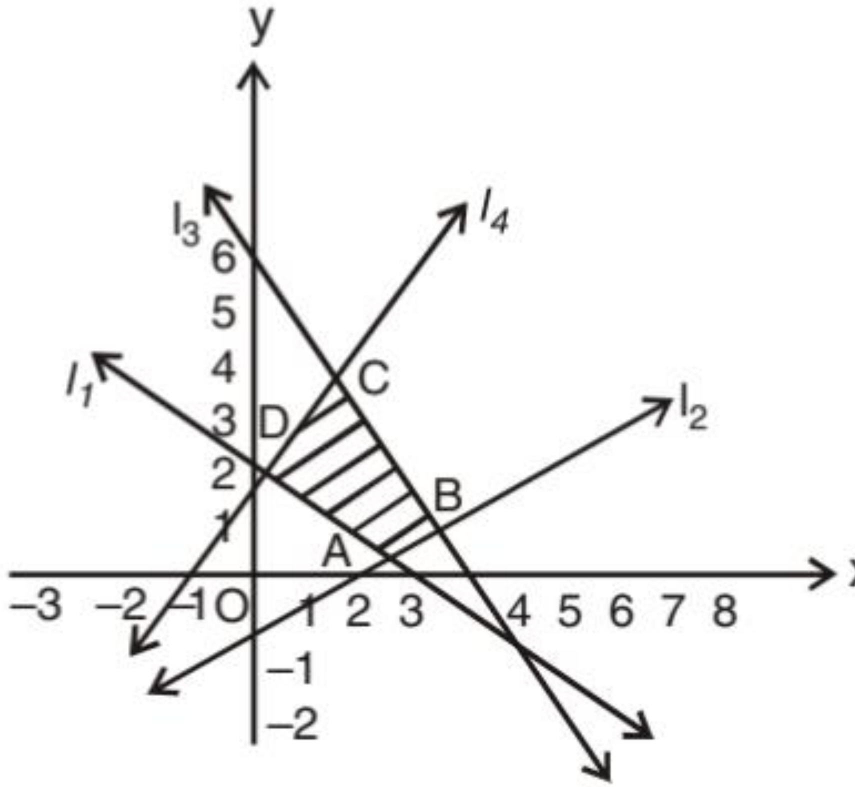
$$-3x+2y=3 \Rightarrow \frac{x}{-1}+\frac{y}{\frac{3}{2}}=1 \qquad ...(4)$$

$$\text{and } x=0, y=0 \qquad ...(5)$$

Clearly the line (1) meets the axes at (3, 0) and (0, 2) the line (2) meets the axes at (2, 0) and (0, – 1) the line (3) meets the axes at (4, 0) and (0, 6) the line (4) meets the axes at (–1, 0) and $\left(0, \frac{3}{2}\right)$. Also $x = 0$ is the y-axis and $y = 0$ is the x-axis.

Plotting the above points and joining them we get the lines l_1, l_2, l_3 and l_4 as shown in fig. The feasible region is the interior of the quadrilateral $ABCD$ as shown in fig. Solving equations (1) and (2) we find that the vertex A is $\left(\frac{18}{7}, \frac{2}{7}\right)$.

Similarly the Coordinates of vertices B, C and D are $\left(\frac{7}{2}, \frac{3}{4}\right)$, $\left(\frac{3}{2}, \frac{15}{4}\right)$ and $\left(\frac{3}{13}, \frac{24}{13}\right)$.



The Cooresponding values of objective function $5x + 2y$ are tabulated below:

Vertex	Co-ordinates	Values of $5x = 2y$
A	$\left(\frac{18}{7}, \frac{2}{7}\right)$	$5 \times \frac{18}{7} + 2 \times \frac{2}{7} = \frac{94}{7} = 13\frac{3}{7}$
B	$\left(\frac{7}{2}, \frac{3}{4}\right)$	$5 \times \frac{7}{2} + 2 \times \frac{3}{4} = \frac{38}{2} = 19$
C	$\left(\frac{3}{2}, \frac{15}{4}\right)$	$5 \times \frac{3}{2} + 2 \times \frac{15}{4} = 15$
D	$\left(\frac{3}{13}, \frac{24}{13}\right)$	$5 \times \frac{3}{13} + 2 \times \frac{24}{13} = \frac{63}{13} = 4\frac{11}{13}$

From above table we observe that maximum value 19 occurs at $B\left(\frac{7}{2}, \frac{3}{4}\right)$ and minimum value occurs $4\frac{11}{13}$ at $D\left(\frac{3}{13}, \frac{24}{13}\right)$.