MATRICES AND DETERMINANTS

Definitions: A system of $m \times n$ numbers (real or complex) arranged in the form of an ordered set of m horizontal lines (called rows) and n vertical lines (called columns) is called an $m \times n$ matrix (read as m by n matrix).

The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1j} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2j} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots\cdots & a_{ij} & \cdots\cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mj} & & a_{mn} \end{bmatrix}$$

Notations: (*i*) Matrices are generally denoted by capital letters of the alphabet *viz*. A, B, C, ..., L, M, ..., X, Y ......

(*ii*) The elements are generally denoted by corresponding small letters *viz.* a_{ij}, b_{ij}, c_{ij} ..., l_{ij}, m_{ij}, ... x_{ij}, y_{ij}, ...

(*iii*) The following notations are used to enclose the elements which constitute a matrix.

[ ], ( ), || ||, { }

Types of matrices

(***i***) **Square matrix:** Any $n \times n$ matrix is called a square matrix of order n (or n-rowed matrix).

In this case, the number of row = the number of columns.

For example. (*a*) $\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ is a square matrix of order 2.

(*b*) $\begin{bmatrix} 1 & 4 & 8 \\ 7 & 3 & 1 \\ 2 & 6 & 5 \end{bmatrix}$ is a square matrix of order 3.

(***ii***) **Rectangular matrix:** Any $m \times n$ matrix, where $m \neq n$ is called a rectangular matrix.

For example, $\begin{bmatrix} 2 & 7 & 8 \\ 3 & 2 & 6 \end{bmatrix}$ is a rectangular matrix.

(iii) **Row matrix:** Any $1 \times n$ matrix is called a row matrix. A row matrix has only one row.

For example. $[6 \quad 2 \quad 8]$ is a row matrix.

(iv) **Column matrix:** Any $m \times 1$ matrix is called a column matrix. A column matrix has only one column.

For example, $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ is a column matrix.

(v) **Principal diagonal:** The line along which the diagonal elements lie is said to be the principal diagonal. It is also called as leading diagonal or simply diagonal. Thus $a_{11}, a_{22} \ldots\ldots a_{ii}$ form the principal diagonal.

For example, If $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -7 & 3 \\ -9 & 0 & 7 \end{bmatrix}$

then the diagonal elements are 1, –7, 7 and principal diagonal is the line on which these lies.

(*vi*) Diagonal matrix: A square matrix $A = [a_{ij}]$ is said to be diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

Thus, it is a square matrix in which all the elements except the diagonal elements are zero.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is a diagonal matrix of order 3.

This is also denoted by symbol [1 3 8]

(*vii*) Diagonal elements of a matrix: An element a_{ij} of the square matrix $A = [a_{ij}]$ is said to be a diagonal element if $i = j$.

Thus $a_{11}, a_{22} \dots a_{ij} \dots$ are diagonal elements.

(*viii*) Identity matrix: A diagonal matrix is said to be an identity matrix if each of its diagonal elements is unity.

This is also known as unit matrix.

Thus $A = [a_{ij}]_{n \times n}$ is called an identity matrix. if (*i*) $a_{ij} = 0$ when $i \neq j$, (*ii*) $a_{ij} = 1$ when $i = j$

The identity matrix of order-n is usually denoted by I_n or simply I.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*ix*) Null matrix: A matrix is said to be a null matrix if each of its elements is zero.

This is also known as zero matrix.

The null matrix of the type $m \times n$ is denoted by $O_{m,n}$ or simply by O.

For example, $O_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$O_{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equality of matrices: Two matrices are said to be equal if and only if, (*i*) They are of the same order.

(*ii*) The elements in the corresponding positions are equal.

Note:

(*i*) If the two matrices A and B are equal, we can write them as A = B.

(*ii*) If the two matrices A and B are not equal, we write them as $A \neq B$.

Operations on matrices: The following three operations on matrices are as follows:

(*i*) Addition of matrices

(*ii*) Multiplication of a martix by a scalar

(*iii*) Multiplication of matrices

Operation I. Addition of matrices. Let A and B be two matrices of the same type $m \times n$ then their sum (denoted by A + B) is defined as the matrix of the same type $m \times n$ obtained by adding the corresponding elements of A and B.

Thus, if $A = \begin{bmatrix} 2 & 3 & -7 \\ 8 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 6 & 9 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 2+1 & 3-2 & -7+3 \\ 8+4 & 2+6 & 9+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -4 \\ 12 & 8 & 18 \end{bmatrix}$

Note: Addition is defined only for matrices which are of the same type.

If two matrices are of the same type, they are said to be conformable for addition.

Difference of two matrices: Let A and B be two matrices of the same type $m \times n$, then their difference (denoted by A – B) is the sum of A and negative of B.

Thus, A – B = A + (–B).

The difference A – B is obtained by subtracting from each element of A the corresponding element of B.

$$\text{Thus} \begin{bmatrix} 2 & 9 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 9-0 \\ 4-3 & 6-1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 5 \end{bmatrix}$$

Properties of Matrix Addition:

(*i*) Matrix addition is commutative: If A and B are any two matrices of the same type $m \times n$, then

$$A + B = B + A$$

(*ii*) Matrix addition is associative: If A, B and C are any three matrices of the same type $m \times n$, then

$$(A + B) + C = A + (B + C)$$

(*iii*) Existence of additive identity: If A and O are the matrices of the same type $m \times n$, then

A + O = O + A = A, where O is the null matrix.

Note: The null matrix O is known as additive identity and is unique.

(*iv*) Existence of additive inverse: If A and –A are the matrices of the same type $m \times n$ such that

$$A + (-A) = O = (-A) + A$$

Then –A is known as additive inverse of A.

(*v*) Cancellation law: If A, B and C are any three matrices of the same type $m \times n$.

Then $A + B = A + C \Rightarrow C = B$

[Left cancellation]

and $B + A = C + A \Rightarrow B = C$

[Right cancellation]

Operation II: Multiplication of a matrix by a scalar. Let A be any $m \times n$ matrix and K be any scalar. Then the $m \times n$ matrix obtained by multiplying every element of matrix A by K is known as scalar multiple of A by K and denoted by KA or AK.

Thus, if $A = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 8 & 5 \end{bmatrix}$

Then,

$$8A = \begin{bmatrix} 8\times 2 & 8\times 3 & 8\times 1 \\ 8\times 6 & 8\times 8 & 8\times 5 \end{bmatrix} = \begin{bmatrix} 16 & 24 & 8 \\ 48 & 64 & 40 \end{bmatrix}$$

Operation III: Multiplication of Matrices.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

Then the product of A and B (written as AB) is given by

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

$= C$ (say)

The entries in the product matrix are obtained as follows:

(*i*) The entry c_{11} in C, which lies in row 1 and column 1, is obtained by multiplying the elements of the first row of A by the corresponding elements of the first column of B. The sum of these products is c_{11}.

(*ii*) The entry c_{12} in C, which lies in row 1 and column 2 is obtained by multiplying the elements in the first row of A by the corresponding elements of second column of B. The sum of these product is c_{12}. Similarly other elements of C can be obtained.

In general. c_{ij} which lies in row i and column j is obtained by multiplying the elements of the ith row of A by the corresponding elements of jth column of B. The sum of these products is c_{ij}.

Note: The product AB of two matrices A and B is defined if and only if the number of columns of A = number of rows of B.

Two such matrices are said to be conformable for multiplication.

Properties of Matrix Multiplication:

(*i*) **Matrix multiplication is not commutative:** If A and B are any two matrices such that AB and BA are defined then $AB \neq BA$. In fact, the following cases may arise:

1. AB may be defined but BA may not be defined.

2. BA may be defined but AB may not be defined.
3. AB and BA may be defined but AB ≠ BA.
4. AB and BA may be defined and AB = BA.

(*ii*) Matrix multiplication is associative: If A, B and C are any three matrices such that A and B are conformable for the product AB , and B and C are conformable for the product BC, then

$$(AB)\,C = A\,(BC)$$

(*iii*) Distributive law: If A, B and C are any three matrices, then

$$A\,(B + C) = AB + AC$$

where the sum and products on both sides of the above relations are defined.

Similarly, $(B + C)\,A = BA + CA$.

(*iv*) Cancellation law does not hold: If A, B and C are matrices such that AB and AC are defined and also AB = AC, then it does not imply

B = C or A cancels out on both sides.

Transpose of a Matrix: The matrix obtained from any given matrix A by interchanging the rows and columns of that matrix is called its transpose and is denoted by A′ or A^t.

For example. If $A=\begin{bmatrix} 2 & 6 & 3 \\ x & y & z \end{bmatrix}$ then $A'=\begin{bmatrix} 2 & x \\ 6 & y \\ 3 & z \end{bmatrix}$

Determinants: Let us eliminate x and y from the equations

$$a_1x + b_1y = 0$$

$$a_2x + b_2y = 0$$

we get on elimination as

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \quad \left[\because \text{ each} = -\frac{y}{x}\right]$$

$$a_1b_2 - a_2b_1$$

which in compact form can be written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

In other words, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ which is a compact form of $a_1b_2 - a_2b_1$ is called a determinant of second order. The number a_1, b_1, a_2, b_2 are called the elements or constituents of the determinant.

Note:

(*i*) A determinant of second order is a function of $2^2 = 4$ elements arranged in two vertical lines called columns and two horizontal lines called rows in the form of a solid.

(*ii*) It has $2! (= 2 \times 1 = 2)$ terms, half of them are positive and half are negative.

(*iii*) Each term in the expansion has one and only one element from each row and each column.

Let us eliminate x, y, z from the equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

Solving the last two equations by cross multiplication

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2} = k(\text{say})$$

$$x = k(b_2c_3 - b_3c_2);\ y = k(c_2a_3 - c_3a_2);\ z = k(a_2b_3 - a_3b_2).$$

Substituting these values of x, y and z in the first equation and cancelling k, we get

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0 \quad \ldots(i)$$

The L.H.S. expression of (i) can be written in compact form as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ which is called the determinant of third order and L.H.S. of (i) is called the expansion or the value of the determinant.

Note:

(i) A determinant of third order is a function of 3^2 (= 9) elements arranged in three vertical lines called columns and three horizontal lines called rows in the form of a solid square.

(*ii*) It has 3! (3 × 2 × 1 = 6) terms, half of them positive and half as negative.

(*iii*) Each term in the expansion has one and only one element from each row and each columns.

Expansion of second order determinant:

$$\begin{matrix} (+) & (-) \end{matrix}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

sign of a product remains unchanged with downward arrow while it changes with upward arrow

$$\begin{matrix} (+) & (-) \end{matrix}$$

$$e.g. \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} = 3 \times 1 - 6 \times 2 = 3 - 12 = 9$$

Expansion of third order determinant: The expansion of third order determinant is explained as

(*i*) Take the element of the first row and first column

$$e.g. \qquad a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(*ii*) Reject all the elements of the row and the column to which a_1 belongs and form the determinant out of the remaining elements. We thus get the determinant of second order

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

The determinant is called the minor of the element a_1, obtain the product of a_1 and its minor. It is written as

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

(*iii*) Take the element of the first row and the second column *i.e.* b_1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(*iv*) Reject all the elements of the row and the column to which b_1 belongs and form the determinant of the left out elements. You get the determinant

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

It is called the minor of the element b_2. Obtain the product of the element b_1 and its minor, we get

$$b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

(*v*) Finally take the element of the first row and the third column *i.e.*, c_1.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(*vi*) Reject all the elements of the row and the column to which c_1 belongs. Obtain the minor of the element c_1 by forming the determinant of the remaining element. The minor of c_1 is

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Find out the product of the element c_1 and its minor and get the product

$$c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(*vii*) The product thus obtained are fixed with plus (+) and minus (–) signs alternately and then find their algebraic sum.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The sum obtained is equal to the value of the original determinant of third order.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note:

(*i*) A determinant can be expanded by any row or by any column.

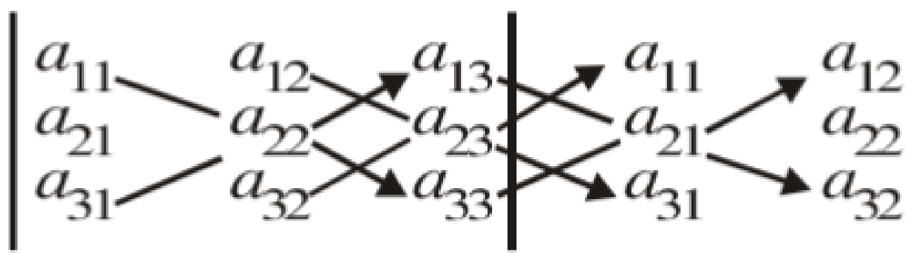
(*ii*) The sign before any term in the expansion is $(-1)^{i+j}$. For determinant of order 3, the signs are

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

(*iii*) Expand from a row or column which contains maximum number of zeros.

Sarrus diagram for expansion of determinants of order 3: This is an easy method for evaluating a third order determinant but it is not applicable to determinants of order higher than 3.

Here repeat the first and second column to the right of the determinant



The term prefixed with +ve sign in the value of the determinant are those which correspond to the elements joined by continuous lines and the term prefixed with –ve sign are those which corresponds to the element joined by dotted lines in the above figure. The value of

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Minors and co-factors

Minor: The determinant obtained from a given determinant by omitting the row and the column in which a particular element lies is called the minor of that element.

Consider the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

To find the minor of c_2. c_2 occurs in the second row and third column. Omitting the second row and third column, the minor of c_2 is

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Co-factor: Co-factors are minors with proper sign and are generally denoted by the corresponding capital letters.

The co-factor of a particular element is $(-1)^{i+j}$ times the determinant obtained from the given determinants by omitting the row and a column in which it lies where i is the number of row and j the number of column, in which that element lies.

e.g., C_2 = co-factor of c_2 in Δ

$$= (-1)^{2+3}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \left[\because (-1)^5 = -1\right]$$

Properties of determinant:

Property I: If each element of a row or a column of a determinant is zero, then the value of the determinant is zero

$$i.e., \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0$$

Property II: The value of a determinant is not altered by changing its rows into columns and columns into rows.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be the given determinant.

Let Δ' be the determinant obtained from Δ by changing its rows into columns and columns into rows.

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Then, $\Delta = \Delta'$

Property III: If two adjacent rows or columns of a determinant are interchanged, the determinant changes in sign but its numerical value is unaltered.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let Δ' be the determinant obtained from Δ by interchanging its 1st and 2nd rows.

$$\Delta' = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

Then, $\Delta' = -\Delta$

Property IV: If two rows or columns of a determinant are identical (same) then the value of the determinant is zero.

$$\Delta = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$$

[Here, 1st and 2nd column are identical]

Property V: If each element of a row or column of a determinant be multiplied by the same factor, then the determinant is multiplied by that factor.

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let Δ' be the determinant obtained from Δ by multiplying the elements of the first column by a constant k.

$$\Delta' = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$$

Then, $\Delta' = k\,\Delta$

Property VI: If each element of a row or column of a determinant can be expressed as the sum of two (or more) terms, then the determinant can also be written as the sum of two (or more) determinants.

$$i.e., \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Property VII: If each element of a row (or column) of a determinant be added or subtracted the equimultiples of the corresponding elements of one or more rows (or columns), the determinant remains unaltered.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ' be the determinant obtained from Δ by adding p times the elements of the second column and subtracting q times the elements of the third column from the first column.

$$\Delta' = \begin{vmatrix} a_1+pb_1-qc_1 & b_1 & c_1 \\ a_2+pb_2-qc_2 & b_2 & c_2 \\ a_3+pb_3-qc_3 & b_3 & c_3 \end{vmatrix}$$

Then, $\Delta' = \Delta$

A consistent system of non-homogeneous linear equations: If a system of linear equations has a solution, we say that the system is consistent. A consistent system of linear equations may have a unique solution or an infinite number of solutions. e.g.,

(A) Let $x + y = 3$ and $x - y = 1$ be a system of two non-homogeneous linear equations which on solving gives the value of $x = 2, y = 1$ and therefore these equations are consistent. The solution $x = 2, y = 1$ is a unique solution.

(B) Let $x + y = 3$ and $3x + 3y = 9$ be another system of linear equations. These equations are satisfied by $x = 0, y = 3$; $x = 1, y = 2$ etc. There are infinite solutions of the given equation.

Inconsistent system: If a system of linear equations has no solution (*i.e.*, if there exists no common values of x, y or x, y, z satisfying the given equations we say the system of linear equations is inconsistent. For example $x + y = 2$ and $x + y = 3$ is a system of inconsistent equations because there exists no values of x and y which satisfy both the equations. [From the two equations equating the two values of $x + y$, we get $2 = 3$ which is not possible].

Crammer's Rule to solve linear equations by determinants: Consider three linear equations in three unknowns.

$$a_1x + b_1y + c_1z = d_1 \quad \text{...}(i)$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{...}(ii)$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{...}(iii)$$

Let D = the determinant of the coefficient of x, y, z in (i), (ii) and (iii)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Replacing the elements of first column by d_1, d_2, d_3.

Let $$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \text{...}(iv)$$

Replacing the elements of second column by d_1, d_2, d_3.

Let $$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Replacing the elements of third column by d_1, d_2, d_3.

Let
$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Substituting the values of d_1, d_2, d_3 from (*i*), (*ii*) and (*iii*) in (*iv*)

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

We have

$$D_1 = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

$$= x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + z\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

(Two columns are same) (Two columns are same)

$$= x\,(D) + y\,(0) + z\,(0) = xD$$

or $xD = D_1$ ...(*v*)

Similarly $yD = D_2$ ...(*vi*)

and $zD = D_3$ ...(*vii*)

If $D \neq 0$, then $x = \dfrac{D_1}{D}, y = \dfrac{D_2}{D}, z = \dfrac{D_3}{D}$

(unique solution)

Note 1: If $D = 0$, $D_1 = 0$, $D_2 = 0$, $D_3 = 0$ then (*v*), (*vi*) and (*vii*) become $x\,(0) = 0$, $y\,(0) = 0$, $z(0) = 0$. These equations are satisfied by all values of x, y and z (an infinite solutions).

Note 2: If $D = 0$ but at least one of D_1, D_2, D_3 is non-zero then the value of at least one of x, y, z tends of infinity and therefore the solution does not exist. Hence, equations are inconsistents.

Singular and non-singular matrix: A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Adjoint of a square matrix: Let $A = [a_{ij}]$ be a square matrix of order n. Then the adjoint of the matrix A is the matrix $B = [b_{ij}]$ where $b_{ij} = a_{ij}$ (where A_{ji} is the cofactor of a_{ij} in the determinant A).

The adjoint of a martix A is also called **Adjugate** of a matrix and is denoted by **adj. A**.

Example : Calculate the adjoint of A where

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$$

Sol : Let the adj $A = B = [b_{ij}]$

$$b_{11} = \text{the cofactor of } a_{11} \text{ in } |A| = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3$$

$$b_{21} = \text{the cofactor of } a_{12} \text{ in } |A| = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9$$

$$b_{31} = \text{the cofactor of } a_{13} \text{ in } |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$b_{12} = \text{the cofactor of } a_{21} \text{ in } |A| = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$$

$$b_{22} = \text{the cofactor of } a_{22} \text{ in } |A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$b_{32} = \text{the cofactor of } a_{23} \text{ in } |A| = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$$

$$b_{13} = \text{the cofacor of } a_{31} \text{ in } |A| = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$b_{23} = \text{the cofactor of } a_{32} \text{ in } |A| = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4$$

$$b_{33} = \text{the cofactor of } a_{33} \text{ in } |A| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\text{adj. } A = B = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} 3 & -4 & 5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{vmatrix}$$

Theorem: If A is an n-rowed square matrix, then A (adj. A) = (adj. A) A = $|A|\ I_n$; where I_n is a n -rowed unit matrix.

Inverse of a square matrix: Let A be a square matrix of order $n \times n$. If there exists another square matrix B of order $n \times n$ such that $AB = BA = I_n$, then B is called inverse or reciprocal of the square matrix A and the matrix A is known as invertible. It is denoted by A^{-1}. So that $AA^{-1} = A^{-1}A = I_n$; obviously $(A^{-1})^{-1} = A$.

Note 1: A rectangular matrix does not possess inverse.

Note 2: If B is the inverse of A, surely A is also inverse of B.

Theorem 1: Inverse of every square matrix if it exists is unique.

Theorem 2: The necessary and sufficient condition for a square matrix A to possess an inverse is that it must be non-singular.

Note: Singular matrices cannot have inverses.

Cor.1. $$A^{-1} = \frac{\text{adj. } A}{|A|}$$

Cor. 2. If $$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{then } A^{-1} = \frac{1}{|A|}\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$$

where A_{ij} is the cofactor of a_{ij} in $|A|$, for $i, j = 1, 2, 3$

Example: Compute the inverse of the matrix

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Sol: Let the given matrix be denoted by A, then

$$|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 6 - 5 = 1 \neq 0$$

Thus A^{-1} exists. Let us calculate the adj. A.
The cofactors of the elements of the first row in $|A|$ are

$$\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \; i.e., 3, -15, 5$$

The cofactors of the elements of the second row in $|A|$ are

$$-\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} \; i.e., -1, 6, -2$$

The cofactors of the elements of the 3rd row in |A| are

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} i.e., 1, -5, 2.$$

$$\text{Adj. } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj. A}}{|A|} = \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix}$$

Working Rule for solving the linear equations:

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Step I: Write down the given system of equations in a single matrix equation AX = B.

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$

Step II: Evaluate $|A|$, i.e., find $|A| = a_1 b_2 - a_2 b_1$

Case I: When $|A| \neq 0$.

Step III: If the matix A is non-singular i.e., $|A| \neq 0$ then A^{-1} exists and find A^{-1}.

Step IV: Pre-multiply both sides of the equation $AX = B$ by A^{-1} so that

We have $A^{-1} AX = A^{-1} B \Rightarrow X = A^{-1} B$

$[\because AA^{-1} = I]$

Step V: Equate the two matrices and get the values of x and y.

Case. II: When $|A| = 0$. In this case A^{-1} does not exist.

There are two possibilities.

(a) When $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

In this case the given system of equation are *inconsistant*.

(b) When $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

In this case the equations are consistent and have an infinite solutions.