SEQUENCE AND SERIES

Sequence: A sequence is a special class of function whose domain is the set of N of all natural numbers and range is any set, i.e., A function $f : N \to S$ is called a sequence. If the range of a sequence is any subset of real numbers, then the sequence is called a real sequence.

Therefore, a real sequence is a function from the set N of natural numbers to the set $S \subset R$ of real numbers.

Different Ways of Describing a Sequence

(*i*) A sequence may be described by listing its first few elements, till we get a rule for writing down the other different members of the sequence, e.g. $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \right\rangle$ is the sequence whose n^{th} term is $\frac{1}{n}$.

(*ii*) Another way of representing the member of the sequence is to specify the rule for its n[th] term, e.g., the sequence $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \right\rangle$ can be rewritten as $\langle f_n \rangle$ where $f_n = \frac{1}{n}$ for all $n \in N$.

Here $f_n = \frac{1}{n}$, gives the rule for the n^{th} term of the sequence.

(*iii*) Lastly, a sequence can also be described by specifying the first term and storing a rule for determining f_n for all $n \geq 1$ in terms of the term f_1, f_2, f_3 ..... e.g., sequence $\langle f_n \rangle$ for which $f_0 = 1$, $f_1 = 2$ and $f_n = \frac{1}{2}\ (f_{n-1} + f_{n-2})$ for all $n \geq 2$ i.e., $\left\langle 1, 1, \frac{8}{2}, \frac{7}{4} \ldots\ldots \right\rangle$

Constant Sequence: The sequence $\langle f_n \rangle$ where $f_n = C \in R$, for all $n \in N$ is called constant sequence. In the case $\langle f_n \rangle = \langle c_1\ c_1\ c_1 \ldots\ldots \rangle$

$\therefore$ Range of $f = \{c\}$ = singleton $c = a$ finite set

It should be noted that

(*i*) It is not necessary that all the terms of the sequence should be distinct.

(*ii*) Care must be taken in distinguishing the range of the sequence and the sequence itself. e.g., the sequence (f_n), where $f_n = (-1)^{n-1}$ for all $n \in \mathrm{N}$ is given by

$$\langle f_n \rangle = \langle 1, \ -1, \ 1, \ -1, \ 1, \ -1 \ldots . \rangle$$

Here, the range of $f = \{1, -1\}$

(*iii*) A sequence by definition is always an infinite set, while the range of the sequence may be finite or infinite, e.g., the sequence $\langle f_n \rangle$ for which $f_n = 1$ for all $n \in \mathrm{N}$.

Thus, $\langle f_n \rangle = \langle 1, \ 1, \ 1, \ \ldots . . \rangle$, where as range of $f = \{1\}$ which is a finite set.

(*iv*) A sequence can also be defined as a succession of terms arranged in a definite order and formed according to a definite law, i.e., the set $\langle f_1, f_2, \ldots f_n \ldots \rangle$ is called a sequence, where $f_1, f_2, \ldots f_n \ldots$ are real numbers arranged in a definite order and formed according to some law.

For example, the set $\langle 1, 4, 9, 16, 25, \ldots \rangle$ is sequence and the law here is
$f_n = n^2$ for all $n \in \mathrm{N}$

Series: A series can be defined as the succession of terms formed and arranged according to some definite law of rule. Let $\langle u_n \rangle$ be a real sequence. The expression of the form $u_1 + u_2 + \ldots + u_n \ldots$ is called a series and is symbolically expressed as $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$. Then real number $u_1, u_2 \ldots$ are known as first, second,... respectively of the series $\sum u_n$.

It should be noted that,
if $\{t_1, t_2, t_3, t_4, \ldots\}$ is a sequence, then
$S = t_1 + t_2 + t_3 + \ldots$ is called the corresponding series.

A sequence is said to be an Arithmetic sequence or Arithmetic progression, if the difference of each term after the first term and the preceding term is constant.

Arithmetic Progression: An arithmetic progression (A.P) is a sequence whose terms increase or decrease by a fixed number. Thus if sequence $\{t_1, t_2, t_3 ...\}$ is such that $t_n - t_{n-1}$ is a constant for all $n \in$ N, it is an A.P. This fixed number is called the common difference of the A.P. For example, 1, 3, 5, 7, 9, ... is an A.P. with common difference = 2.

$$[\because\ 3 - 1 = 5 - 3 = 7 - 5 = 9 - 7 = ... = 2]$$

The standard A.P. is $a, (a + d), (a + 2d), (a + 3d)...$ where a is the first term and d is the common difference.

***n*th term of an A.P.:** If a is the first term and d the common difference of an A.P., then its n^{th} term t_n is given by $t_n = a + (n - 1)d$.

Sum of *n* terms of an A.P.: If a is the first term and d the common difference of an A.P., then sum of n terms (denoted by S_n) is given by

$$S_n = \frac{n}{2}\ [2a + (n - 1)d] = \frac{n}{2}\ (a + l),$$ where l is the last term.

Arithmetic Mean: Arithmetic mean (A.M.) of any two numbers a and b is given by $(a + b)/2$, i.e., the arithmetic mean between two given numbers is equal to half of their sum. If $x_1, x_2, \dots x_n$ are n numbers, then their A.M. is given by $(x_1 + x_2 + \dots + x_n)/n$

***n* arithmetic means:** The numbers $A_1, A_2, \dots A_n$ are said to be n arithmetic means between a and b if

$$a, A_1, A_2, A_3, \dots A_n, b \text{ are an A.P.}$$

Hence, $\quad b = t_{n+2} = a + (n+1)\,d.$

$$\therefore \qquad d = \frac{b-a}{n+1} \text{ and hence, } A_1 = a + \frac{b-a}{n+1}, \dots$$

$$A_r = a + \frac{r(b-a)}{n+1}$$

Important Deductions:

(*i*) If a fixed number is added or subtracted to each term of a given A.P., then the resulting series is also an A.P. and its common difference remains the same.

(*ii*) If each term of an A.P. is multiplied by a fixed constant or divided by a fixed non-zero constant, then the resulting series is also an A.P.

For example, an A.P. is multiplied by the same number, i.e., $ak, (a+d)\,k, (a+2d)\,k \dots$ then the resulting sequence is also an A.P.

(*iii*) If $x_1 + x_2 + x_3 + \ldots$ and $y_1 + y_2 + y_3 + \ldots$ are two A.P., then $x_1 \pm y_1, x_2 \pm y_2, x_3 \pm y_3 + \ldots$ are also in A.P.

(*iv*) If the sum of three numbers in A.P. is given; take them as $a - d, a, a + d$.

(*v*) If the sum of four numbers in A.P. is given; take them as $a - 3d, a - d, a + d, a + 3d$.

(*vi*) If the sum of the five numbers in A.P. is given; take them as $a - 2d, a - d, a, a + d, a + 2d$.

Geometric Progression (G.P.): A sequence is said to be a G.P. or G.S. if the ratio of any term, after the first term, to its proceeding term is constant. This constant ratio is called the common ratio of the G.P. and is usually denoted by r.

It follows from the definition that no term of a G.P. can be zero.

The standard G.P. is

$$a + ar + ar^2 + ar^3 + \ldots \left[\because \frac{ar}{a} = \frac{ar^2}{ar} = \ldots = r\right]$$

where a is the first term of a G.P. and r is the common ratio of G.P.

Note: It should be noted that

(a) for $r = 1$, the G.P. is $a + a + a + \ldots + a$ (n times) and its sum is $S_n = na$ (If $r = 1$);

(b) If we multiply any term of a G.P. by a common ratio, we get the next following term and if we divide any term of the G.P. by the common ratio, we get the proceeding term.

***n*th term of a G.P.:** If a is the first term and r is common ratio of a G.P., then n^{th} term

$$= t_n = ar^{n-1}$$

Sum of *n* terms of a G.P.: If a is the first term and r is common ratio of a G.P., then sum of n terms, denoted by S_n is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \text{ if } r < 1$$

Sum of Infinite G.P.: If $-1 < r < 1$ i.e., $|r| < 1$, then the sum of the infinite G.P.

$$a + ar + ar^2 + ... = a/(1 - r).$$

Geometric mean (G.M.): If a, $b > 0$, then the geometric mean of a and $b = \sqrt{ab}$.

If a_1, a_2 ..., $a_n > 0$, then their geometric mean is given by $(a_1, a_2 ... a_n)^{1/n}$.

If n geometric mean $g_1, g_2, ... g_n$ are to be inserted between two positive real numbers a and b then $a, g_1, g_2, ..., g_n, b$ are in G.P.

so, $b = ar^{n+1}$ *i.e.,* $r = (b/a)^{1/(n+1)}$ and

then $g_1 = ar, g_2 = ar^2, ..., g_n = ar^n$

Some Important Deductions:

(*i*) If each term of a G.P. is multiplied or divided by some fixed non-zero number, then resulting sequence is also a G.P.

(*ii*) If x_2, x_2, x_3 ... and y_1, y_2, y_3, ... are two G.P. then

x_1y_1, x_2y_2, x_3y_3, ... and $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}$, ... are also in G.P.

(*iii*) If x_1, x_2, x_3 ... is a G.P. of positive terms, then $\log x_1, \log x_2, \log x_3$, ... is an A.P. and vice versa.

(*iv*) Three terms of a G.P. can be taken as a/r, a, ar and four terms in G.P. as a/r^3, a/r, ar, ar^3. This presentation is useful if product of terms is involved in the problem. In other problems, terms should be taken as a, ar, ar^2 ... .

Harmonic Progression (H.P.): The sequence x_1, x_2, x_3 ..., x_n ..., where $x_n \neq 0$ for each n, is said to be in H.P. if the sequence $1/x_1$, $1/x_2$, $1/x_3$ ... is an A.P. It should be noted that no term of H.P. can be zero.

***n*th term of a H.P.:** If x_1, x_2, x_3, ... is a H.P., then the corresponding A.P. is $\frac{1}{\ }, \frac{1}{\ }$ ... .

Hence, $a = \frac{1}{x_1}$ and $d = \frac{1}{x_2} - \frac{1}{x_1}$, then nth term of H.P. $(x_n) = \frac{1}{a+(n-1)d}$

Harmonic Mean (H.M.): If a and b are two non-zero numbers, then the harmonic mean of a and b, denoted by H is given by $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$ or $H = \frac{2ab}{a+b}$. If $a_1, a_2, a_3, ..., a_n$ be n non-zero numbers, then their harmonic mean H is given by

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$$

If $H_1, H_2, \ldots H_n$ are n harmonic means between two non-zero numbers a and b, then

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \ldots \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

Arithmetico-Geometric Series (A.G.S.): A series in which each term is obtained by multiplying the corresponding terms of an arithmetic and geometric series is defined as an Arithmetico-Geometric series.

Let $a + (a + d) + (a + 2d) + ... + [a + (n - 1)\, d]$ is a standard A.P.

and $1 + r + r^2 + ... + r^{n-1} + ...$ is a standard G.P.

Then Arithmetico-Geometric series

$$= a + (a + d)\, r + (a + 2d)\, r^2 + ... + \left(a + \overline{n-1}d\right) r^{n-1} + ...$$

***n*th term of A.G.S.:** The nth term of A.G.S. is $\{a + (n - 1)\, d\}\, r^{n-1}$

Sum of *n* terms of A.G.S.: The sum of n terms of this series

$$S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{\left[a+(n-1)d\right]r^n}{1-r}$$

Sum to infinity $S = \dfrac{a}{1-r} - \dfrac{dr}{(1-r)^2}$

Formulae for Σn, Σn^2, Σn^3

(*i*) $\Sigma n = \sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{1}{2}\, n\,(n + 1)$

(*ii*) $\Sigma n^2 = \sum_{r=1}^{n} r^2 = \frac{1}{6} n (n+1)(2n+1)$

(*iii*) $\Sigma n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{1}{2} n(n+1)\right]^2$

Connection between A.M., G.M. and H.M.: If A, G and H are respectively the A.M., G.M., and H.M. of two numbers a and b then

(*i*) $AH = G^2$ (*ii*) $A \geq G \geq H$