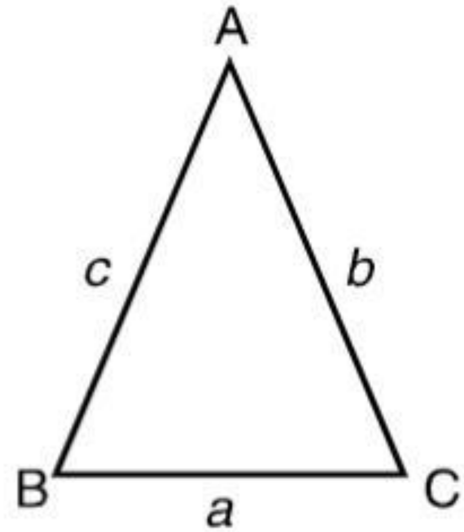
SOLUTION OF TRIANGLES

1. **Relation between Sides and Angles of a Triangle:** In any triangle ABC, the capital letters A, B, C denote the angles and the small letters a, b, c denote the side opposite to the angles A, B, C respectively.



2. **Sine Rule (Law of Sines):** In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

[Any triangle, the sides are proportional to the sines of the opposite angles.]
The above formula can also be put as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

I. To express the sides of a triangle in terms of angles
From sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{k (say)}$$

$\therefore$ $a = \text{k} \sin A$, $b = \text{k} \sin B$, $c = \text{k} \sin C$.

II. To express the sines of the angles of a triangle in terms of sides

$$a = \text{k} \sin A,\ b = \text{k} \sin B,$$
$$c = \text{k} \sin C$$

$$\therefore \quad \sin A = \frac{a}{\text{k}},\ \sin B = \frac{b}{\text{k}},\ \sin c = \frac{c}{\text{k}}$$

3. **Tangent Rule (Napier's Analogy):** In any triangle ABC, $\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$;

$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$;

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

4. **The cosine Rule (Law of cosine):** In any triangle ABC,

$$\cos A = (b^2 + c^2 - a^2)/2bc$$
$$\cos B = (c^2 + a^2 - b^2)/2ca$$
$$\cos C = (a^2 + b^2 - c^2)/2ab$$

Here,
$$c^2 = a^2 + b^2 - 2\,ab \cos C$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$

5. **Projection Rule:** In any triangle ABC

$$a = b \cos C + c \cos B$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A$$

6. **Area of a Triangle:** The area of a ΔABC, denoted by Δ is given by the following formulae:

(i) $\Delta = \frac{1}{2}\ bc \sin A = \frac{1}{2}\ ca \sin B = \frac{1}{2}$

$ab \sin C.$

(ii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$,

where $s = (a + b + c)/2$

(iii) $$\Delta = \frac{b^2 \sin C \sin A}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

$$= \frac{a^2 \sin C \sin B}{2 \sin A}$$

7. Trigonometrical Ratios of the Half Angles of a Triangle

A. (i) $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

(ii) $$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

(iii) $$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

B. (i) $$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

(ii) $$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$(iii)\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

C. $$(i)\ \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$(ii)\ \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$(iii)\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

D. $$(i)\ \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{2\Delta}{bc}$$

$$\sin B = \frac{2\Delta}{ca}$$

$$\sin C = \frac{2\Delta}{ab}$$

8. **Circumcircle of a triangle:** A circle passing through the vertices of a triangle is called the circumcircle of the triangle.

The centre of the circumcircle is called the circumcentre of the triangle and it is the point of intersection of the perpendicular bisectors of the sides of the triangle.

The radius of the circumcircle is called the circumradius of the triangle and is usually denoted by R and is given by the following formulae.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

9. **Incircle of a Triangle:** A circle which can be inscribed within the triangle so as to touch all the three sides is called the incircle of the triangle.

The centre of the incircle is called the incentre of the triangle and it is the point of intersection of the internal bisectors of the angles of the triangle.

The radius of the incircle is called the in-radius of the triangle and is usually denoted by r and is given by the following formulae:

$$r = \Delta/s = (s-a)\tan\frac{1}{2}A$$

$$= (s-b)\tan\frac{1}{2}B = (s-c)\tan\frac{1}{2}C.$$

$$r = \frac{a\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}$$

$$= \frac{b\sin\frac{1}{2}A\sin\frac{1}{2}C}{\cos\frac{1}{2}B}$$

$$= \frac{C\sin\frac{1}{2}B\sin\frac{1}{2}A}{\cos\frac{1}{2}C}$$

$$r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C.$$

Escribed Circles (Ex-Circle) of a triangle: The circle which touches the side BC and the other two sides AB and AC produced of a triangle ABC is called the escribed or ex-circle opposite to the angle A.

Similarly we can define the escribed circle opposite to the angles B and C.

The radii of the escribed circles opposite to the angles A, B and C are called the ex-radii and are usually denoted by r_1, r_2, r_3 respectively and are given by the following formulae–

(i) $r_1 = s \tan \frac{1}{2} A$; $r_2 = s \tan \frac{1}{2} B$;

$r_3 = s \tan \frac{1}{2} C$.

(ii) $$r_1 = \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A};$$

$$r_2 = \frac{b \cos \frac{1}{2} C \cos \frac{1}{2} A}{\cos \frac{1}{2} B};$$

$$r_3 = \frac{c \cos \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} C}$$

(iii) $r_1 = 4R \sin \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$,

$$r_2 = 4\text{R} \cos \frac{1}{2} \text{A} \sin \frac{1}{2} \text{B} \cos \frac{1}{2} \text{C},$$

$$r_3 = 4\text{R} \cos \frac{1}{2} \text{A} \cos \frac{1}{2} \text{B} \sin \frac{1}{2} \text{C}.$$

Cyclic Quadrilateral: A quadrilateral ABCD is said to be a cyclic quadrilateral if there a circle passing through all its four vertices A, B, C and D.

(i) Area of a cyclic quadrilateral ABCD

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where AB = a, BC = b, CD = c, DA = d and $2s = a + b + c + d$

(ii) Circumradius of a cyclic quadrilateral

$$\text{R} = \frac{1}{4} \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

Regular Polygon: A polygon is said to be a regular polygon if all its sides are equal and its angles are equal.

(i) *Circumscribed Circle:* The circle passing through all the vertices of a regular polygon is called its circumscribed circle.

If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle is given by

$$R = \frac{a}{2\sin(\pi/n)} = \frac{a}{2} \operatorname{cosec}(\pi/n)$$

(ii) Inscribed Circle: The circle which can be inscribed within the regular polygon so as to touch all its sides is called its inscribed circle.

The radius r of inscribed circle of a regular polygon of n sides, each of length a is given by

$$r = \frac{a}{2} \cot\left(\frac{\pi}{n}\right)$$

(iii) Area of a regular polygon: The area, Δ of a regular polygon of n sides, each of length a is given by

$$\Delta = \frac{1}{4} na^2 \cot(\pi/n)$$

$$= \frac{1}{2} nR^2 \sin(2\pi/n)$$

$$= nr^2 \tan(\pi/n)$$