VECTOR ALGEBRA

Scalar quantity: Quantities which have magnitude but no direction are called scalar quantities or simply scalars. For example, mass, speed, time etc.

Vector quantities: Quantities which have magnitude as well as direction are called vector quantities or simply vectors. For example, force, velocity etc.

Directed lines segment: A line segment (portion of line) which is assigned a definite direction by specifying its initial and terminal points is called a directed line segment. The directed lined segment AB has

(*a*) a definite length

(*b*) a definite direction, from A to B, indicated by the arrow head.

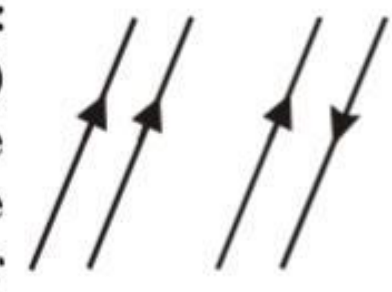
Representation of a vector: A directed line segment represents a vector. A vector whose magnitude is proportional to the length AB and whose direction is from A to B and is denoted by $\vec{AB}$ where A is called the initial point and B is the terminal point of the vector $\vec{AB}$. The magnitude of the vector $\vec{AB}$ is denoted by $\left|\vec{AB}\right|$ which is read as modulus of $\vec{AB}$ or simply $\vec{AB}$.

$\left|\vec{AB}\right| = AB.$

If the directed line segment AB is a part of the line AB, then the line l is called the support of vector $\vec{AB}$.

Type of vectors:

(*i*) **Like and unlike vectors:** Two (or more than two) vectors are said to be like vectors if they have the same direction (no matter what their magnitudes are). Unlikely vector have opposite directions.



(*ii*) Null vector or zero vector : A vector whose magnitude is 0 is called a null or zero vector and is represented by $\vec{O}$. The initial and terminal points of a zero vectors are co-incident and its direction is arbitrary. Thus,

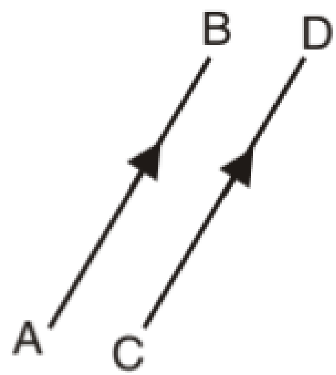
$\vec{AA} = \vec{BB} = \ldots\ldots = \vec{O}$ and $\left|\vec{O}\right| = 0$.

A non-zero vector is called proper vector.

(*iii*) Equal vectors : Two vectors are said to be equal if they have same magnitude and same direction. Thus, $\vec{AB} = \vec{CD}$

if (*a*) $|\vec{AB}| = |\vec{CD}|$ *i.e.* AB = CD

(*b*) $\vec{AB}$ and $\vec{CD}$ are like vectors.



(*iv*) A unit vector : A vector whose magnitude is unity (one) is called a unit vector. If $\vec{a}$ is a unit vector then $|\vec{a}| = a = 1$. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ and read as a-cap. The unit vectors in the positive directions of the axes of x, y and z are denoted by $\hat{i}, \hat{j}, \hat{k}$.

(v) Coplaner Vectors: Two or more vectors are said to be coplaner if they are parallel to the same plane.

(*vi*) Collinear vectors : Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitude.

(*vii*) Negative vectors : Two or more vectors are called negatives of each other if they have same magnitude but opposite direction. Negative of $\vec{a}$ is denoted by $-\vec{a}$.

(*viii*) Localised vectors : A vector having a fixed initial points is called a localised vectors.

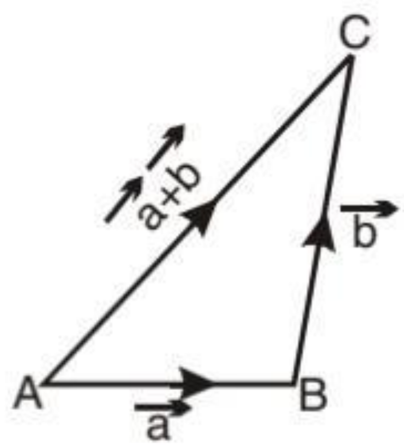
(*ix*) Co-initial vector : Two or more vectors are said to be co-initial vectors if they have the same initial point. For example $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are co-initial vectors with initial point A.

(x) Free vectors : Vectors whose directions and magnitudes are known but the initial point and the support are not known are called free vectors.

Addition of two vectors $\vec{a}$ and $\vec{b}$

(a) *Sum of two vectors by: Triangle Law*

$$\vec{AB} + \vec{BC} = \vec{AC}$$

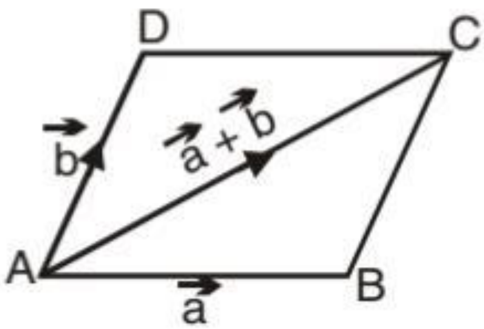


In words : If two vectors are represented by the two sides of a triangle taken in order, then their sum is represented by the third side of the triangle taken in opposite direction.

Note : Two vectors can be added by triangle law only if the terminal points of the directed segment representing one vector is the same as the initial point of the directed segment representing the second vector.

(*b*) *Sum of two vectors by parallelogram:*

$$\vec{AB} + \vec{AC} = \vec{AD}$$



Note : These two methods of addition namely triangle and parallelogram law are identical.

(*c*) *Difference of two vectors :* The difference of two vectors $\vec{a}$ and $\vec{b}$ is defined as the addition of one to the negative of the other.

Thus, $\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$.

Properties of addition of vectors

1. Vector addition is cumulative i.e. if $\vec{a}$ and $\vec{b}$ are two vectors, then $\vec{a}+\vec{b} = \vec{b}+\vec{a}$.
2. Vector addition is associative i.e. for any three vectors $\vec{a}, \vec{b}, \vec{c}$.

   $\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}.$
3. Zero vector is the additive identity i.e from any vector $\vec{a}, \vec{a}+\vec{0}=\vec{a}$ where $\vec{0}$ is the null vector.
4. *Additive inverse.* For any vector $\vec{a}$, there exists the vector $-\vec{a}$ such that $\vec{a}+\left(-\vec{a}\right)=0.$

Properties of multiplication of vector by a scalar

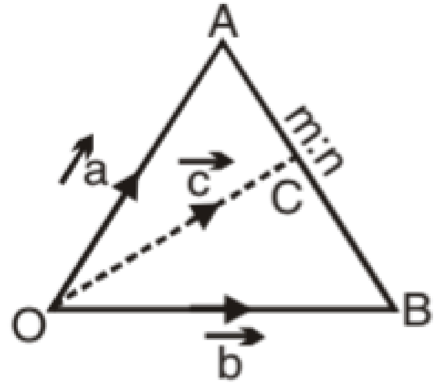
1. *Associative law.* If $\vec{a}$ any vector and m, n are any scalar, then

   $m(n\vec{a})=(mn)\vec{a}$
2. *Distributive law.* If $\vec{a}$ is any vector and k, l are any scalars, then

   $(k+l)\vec{a}=k\vec{a}+l\vec{a}.$
3. *Distributive law.* If $\vec{a}$ and $\vec{b}$ are any two vectors and k is a scalar, then

$$k(\vec{a}+\vec{b}) = k\vec{a}+k\vec{b}$$

Section formula : If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B then the point C which divides AB in the ratio of m ; n where m and n are the positive real numbers has the position vector.



$$\vec{c} = \frac{n\vec{a}+m\vec{b}}{m+n}$$

Cor. Mid-point formula. If $m = n$, then C the mid-point of AB, $\vec{c} = \frac{\vec{a}+\vec{b}}{2}$

Linear combination :

(*i*) A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}\ldots$ if there exists scalars x, y, z ... such that $\vec{r} = x\vec{a}+y\vec{b}+z\vec{c}+\ldots$

(*ii*) **Linear dependent:** A system of vectors, $\vec{a_1}, \vec{a_2} \ldots \vec{a_n}$ is said to be linearly dependent if there exists scalars $x_1, x_2 \ldots x_n$ (not all zero) such that $x_1 \vec{a_1}, x_2 \vec{a_2} + \ldots x_n \vec{a_n} = 0$

Linear independent: A system of vectors $\vec{a_1}$, $\vec{a_2}$ ... $\vec{a_n}$ is said to be linearly dependent if there exist scalar $x_1, x_2 \ldots x_n$ (all zero) such that $x_1 \vec{a_1} + x_2 \vec{a_2} + \ldots x_n \vec{a_n} = 0.$

Remark: $\vec{r}$ and $x\vec{r}$ are collinear, where x is a scalar, $x\vec{a} + y\vec{b}$ represents a vector coplanar with vectors $\vec{a}$ and $\vec{b}$ where x, y are scalars.

Theorem1. If $\vec{a}, \vec{b}$ be two non-zero, non-collinear vectors and x, y are two scalars such that $x\vec{a} + y\vec{b} = 0$, then $x = 0, y = 0$.

Theorem 2. If $\vec{a}, \vec{b}, \vec{c}$ be three non-zero, non-coplanar vectors and x, y, z are three scalars such that, $x\vec{a} + y\vec{b} + z\vec{c} = 0$, then $x = 0, y = 0, z = 0$.

Theorem 3. *Resolution of a vector in terms of coplanar vectors.* If $\vec{a}, \vec{b}$ be two given non-collinear vectors, then every vector $\vec{r}$ can be expressed uniquely as a linear combination $x\vec{a} + y\vec{b}$; x, y being scalars.

Theorem 4. *Non-coplanar vector*

If $\vec{a}, \vec{b}, \vec{c}$ be three given non-coplanar vectors, then any vector $\vec{r}$ can be expressed uniquely as a linear combination, $x\vec{a} + y\vec{b} + z\vec{c}$, x, y, z being scalars.

Unit vector is that vector whose magnitude is unity we denote unit vectors along OX, OY, OZ by $\vec{i}, \vec{j}, \vec{k}$ respectively.

Product of two vectors: The product of two vectors is defined in two ways: (*i*) Scalar product (*ii*) Vector product.

(*i*) Scalar product of two vectors is always a scalar quantity.

(*ii*) Vector product of two vectors is always a vector quantity.

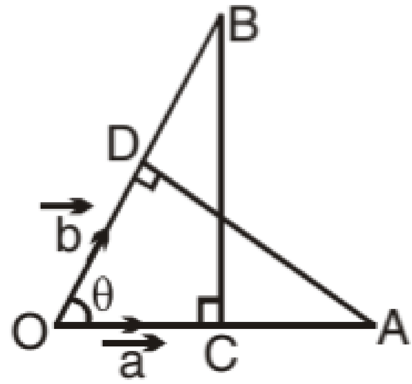
Scalar product: The scalar product of two vectors $\vec{a}$ and $\vec{b}$ with magnitude a and b respectively is defined by the real number $|\vec{a}|\ |\vec{b}| \cos\theta$, where θ is the angle between the direction of a and b. Here, θ is restricted to the interval $0 \le \theta \le \pi$.

It makes no difference whether θ or –θ is chosen as cos θ = cos (–θ).

Notation: The scalar product of two vectors is written as $\vec{a}\ \vec{b}$ or $\left(\vec{a}.\ \vec{b}\right)$, $\vec{a}\ .\ \vec{b}$ is read as $\vec{a}$ dot $\vec{b}$. Because of this the scalar product is sometimes called the dot product.

Aid to memory: $\vec{a}.\ \vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$, which is +ve, –ve or zero according as θ is acute, obtuse or a right angle.

Geometrical interpretation: The scalar product of two vector is the product of the modulus of either vector and the scalar component of the other in its direction.



Aid to memory: If $\vec{a}$ and $\vec{b}$ are two vectors, then the projection of b on the direction of $\vec{a} = \dfrac{\vec{a}.\ \vec{b}}{|\vec{a}|}$.

Condition of perpendicularity:

Theorem. If $\vec{a}$ and $\vec{b} = 0$, are perpendicular vectors, then $\vec{a}.\ \vec{b} = 0$

Conversely: If $\vec{a}.\ \vec{b} = 0$ then either at least one of the two vectors is a zero vector or two vectors are perpendicular.

Some important results:

1. *When the two vectors are like parallel,*

   Here $\theta = 0°$ $\therefore$ $\cos\theta = \cos 0° = 1$.

   $\therefore\ \vec{a}.\ \vec{b} = |\vec{a}|\,.\,|\vec{b}|\cos\theta = ab\ (1) = ab.$

2. When the two vectors unlike parallel,

   Here $\theta = \pi$

   $\therefore\ \cos\theta\ = \cos\pi = -1$

   $\therefore\ \vec{a}.\ \vec{b} = |\vec{a}|\,.\,|\vec{b}|\cos\theta = ab\ (-1) = -ab$

3. $\vec{a}.\vec{a} = a^2.$

4. Since $\vec{i},\ \vec{j},\ \vec{k}$ are vectors perpendicular to each other,

   $\therefore\ \vec{i}.\ \vec{j} = \vec{j}\ .\vec{k} = \vec{k}.\ \vec{i} = 0$

   and $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = 1.$

Aid to memory: These results can be remembered with the help of the following table.

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
i	1	0	0
j	0	1	0
k	0	0	1

Properties of scalar or dot products

Property 1. Comulative law. If $\vec{a}, \vec{b}$ be any two vectors, then $\vec{a}.\vec{b} = \vec{b}\,.\,\vec{a}$.

Property 2. Associative law does not hold. If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $\left(\vec{a}.\,\vec{b}\right).\vec{c} \neq \vec{a}.\left(\vec{b}.\vec{c}\right)$.

Property 3. If x is any scalar then

$$(x\,\vec{a}).\,\vec{b} = x\left(\vec{a}\,.\,\vec{b}\right) = \vec{a}\,.\left(x\vec{b}\right).$$

Property 4. Distributive law. If $\vec{a}.\,\vec{b}, \vec{c}$ are any three vectors, then

$$\vec{a}\,.\left(\vec{b}+\vec{c}\right) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$$

Cor. $\vec{a}\,.\left(\vec{b}.-\vec{c}\right) = \vec{a}\,.\left[\vec{b}+(-\vec{c})\right] = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,(-\vec{c})$

$$= \vec{a}\,.\,\vec{b}\,. - \vec{a}\,.\,\vec{c}$$

Property 5. $\vec{a}\,.\,\vec{a} \geq 0$ where a is any vector.

Theorem. If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, then $\left(\vec{a} \,.\, \vec{b}\right) = (a_1b_1, a_2b_2, a_3b_3)$.

Angle between two vectors: If θ be the angle between two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Cor. *Condition of parallelism*

$$\frac{a_1}{b_1} = \frac{a_2}{a_2} = \frac{a_3}{b_3}.$$

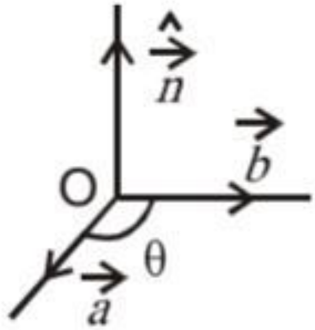
Cor. 2 *Condition of perpendicularity*

$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

Work done by a force : A force acting on a particle is said to do work when the particle is displaced in a direction which is not perpendicular to the direction of the force. The work done is scalar quantity and its measure is defined to the product of force and displacement. Thus $\vec{F}, \vec{d}$ be vectors representing the force and the displacement respectively inclined at an angle θ, the measure of the work done is $\overrightarrow{\mathrm{F}d}\cos\theta = \vec{\mathrm{F}}.\vec{d}$.

Note: Work done is zero if $\vec{d}$ is perpendicular to $\vec{F}$ because in this case, $\cos\theta = \cos\frac{\pi}{2} = 0$.

Vector product: The vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector whose magnitude is $|\vec{a}|\ |\vec{b}|$ $\sin\theta$, where θ is the angle between two direction is that of a unit $\hat{\vec{n}}$ perpendicular to both $\vec{a}$ and $\vec{b}$.



Notation: The vector product of two vectors is written as $\vec{a} \times \vec{b}$ or $[\vec{a} \times \vec{b}]$

$\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$, because this vector product is sometimes called the the cross product.

Remember: $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\, \vec{n}$ where $\vec{a}, \vec{b}, \hat{\vec{n}}$ form a right handed system.

Some important results

1. *When two vectors are parallel,*
   Here $\theta = 0°$.
   $\therefore\ \sin\theta = 0$

$\vec{a} \times \vec{b}, = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{\vec{n}} = \vec{0}$

Particular case. $\vec{a} \times \vec{a} = \vec{0},\ \vec{b} \times \vec{b} = \vec{0}.$

Cor. When $\vec{a} \times \vec{b} = 0$ then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel vectors.

2. *When two vectors are perpendicular.*

   Here $\theta = 90°$ $\qquad \therefore \sin\theta = 1$

   $\therefore\ \vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{\vec{n}} = |\vec{a}|\ |\vec{b}|\ \hat{\vec{n}}$

3. (*a*) *When* $\vec{a}$ *and* $\vec{b}$ *are unit vectors.*

   Here $\vec{a} \times \vec{b} = (1) \sin\theta\ \hat{\vec{n}}$

   $\therefore\ |\vec{a} \times \vec{b}| = \sin\theta.$

   (*b*) When $\vec{a}$ *and* $\vec{b}$ *are not unit vectors.* Here

   $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{\vec{n}}$

   $\therefore |\vec{a}\ \vec{b}| = |\vec{a}|\ |\vec{b}| \sin\theta \Rightarrow \sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\ |\vec{b}|}$

4. *Vector products of unit vectors.* $\vec{i}, \vec{j}, \vec{k}$. Since $\vec{i}, \vec{j}, \vec{k}$ form a right handed system of mutually perpendicular vectors.

$\therefore \quad \vec{i} \times \vec{j}$ is a vector having modulus unity and direction parallel to k.

$$\therefore \quad \vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$$

$$\vec{i} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$$

$$\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

These results can be remembered with the help of following table

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Properties of vector or cross product

Property 1. *Commulative law.* It does not hold

If $\vec{a}, \vec{b}$ are vectors, then $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

Another form $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Property 2.

(a) *Vector product is associative w.r.t. a scalar*

i.e. $(m\vec{a}) \times \vec{b} = \vec{m}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(m\,\vec{b}\right)$, *where m is any scalar.*

(b) *Vector product is not associative w.r.t. to vector.*

i.e. $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$.

Property 1. *Vectro product in terms of rectangular components of vectors*

If $\quad \vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}$

$$\vec{b} = b_1\,\vec{i} + b_2\,\vec{j} + b_3\,\vec{k}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Property 4. Distributive law. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$