

Class 12 maths [Application of Derivatives]

Rate of change (परिवर्तन की दर)

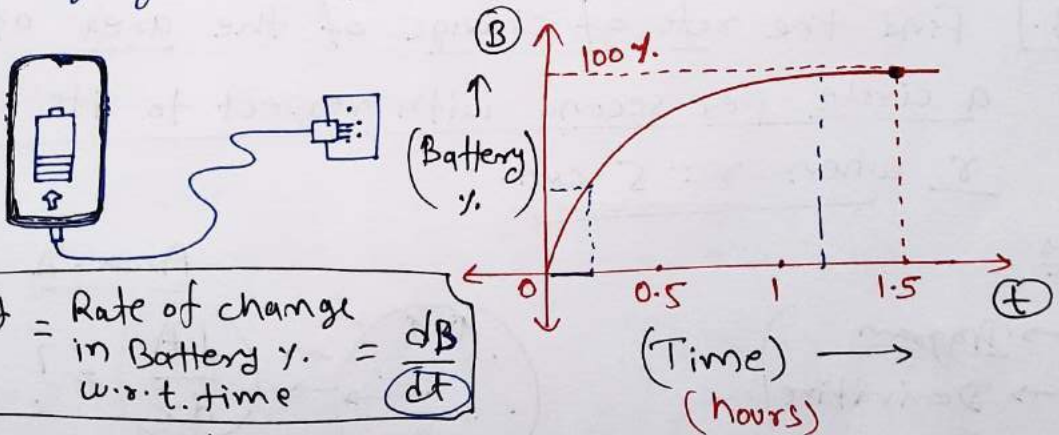
$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) = \text{rate of change of } y \text{ with respect to } x$$

$$\star \left. \frac{dy}{dx} \right|_{x=x_0} = f'(x_0) = \text{rate of change of } y \text{ with respect to } x \text{ at } x=x_0.$$

Instantaneous / तात्क्षणिक
(at some instant) / (किसी क्षण पर)
(किसी पल में)

e.g. Charging Speed of a Mobile Phone.



Charging speed = Rate of change in Battery % w.r.t. time = $\frac{dB}{dt}$

Charging speed in starting > Charging speed in the end

$$\left. \frac{dB}{dt} \right|_{t=0.5} > \left. \frac{dB}{dt} \right|_{t=1}$$

Questions on Rate of Change

How to Deal?

order

- D → Diagram
- D → Derivatives (Given/asked)
- F → Formula
- D → Differentiation
- C → Constants.

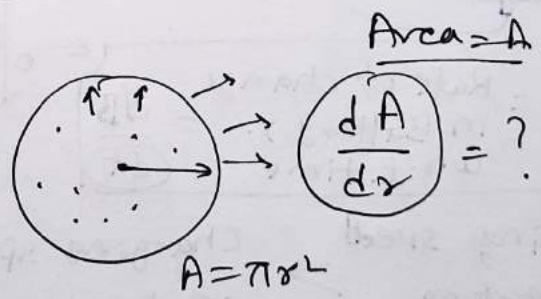
Note:

$\frac{dy}{dx} = \oplus ve$	⇒	$x \uparrow$	$y \uparrow$
$\frac{dy}{dx} = \ominus ve$	⇒	$x \uparrow$	$y \downarrow$

e.g.] Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Ans.

- D → Diagram
- D → Derivative
- F → Formula
- D → Diff.
- C → Constant



by Differentiating w.r.t. r

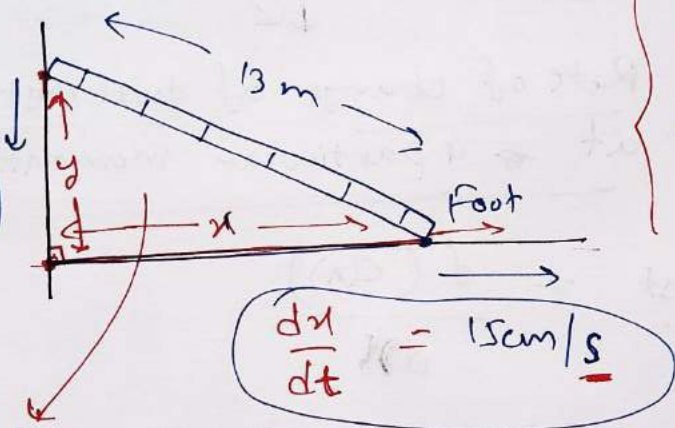
$$\frac{dA}{dr} = \pi(2r) \quad (r = 5 \text{ cm})$$

$$\left. \frac{dA}{dr} \right|_{r=5} = 10\pi$$

e.g. A ladder 13 m long is leaning against a well. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 15 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 12 m away from the wall?

Ans.

$$\frac{dy}{dt} = ?$$



- D → Diagram ✓
- D → Derivatives ✓
- F → Formula ✓
- D → Diffⁿ ✓
- C → Constants ✓

$$\frac{dx}{dt} = 15 \text{ cm/s}$$

Pythagoras Theorem:

$$x^2 + y^2 = 13^2$$

by Diff. w.r.t. 't'

$$\Rightarrow x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = - \left(\frac{x \cdot \frac{dx}{dt}}{y} \right) = - \frac{(12 \times 15)}{5} = -36 \text{ cm/s}$$

$$x = 12 \text{ m}$$

$$x^2 + y^2 = 13^2$$

$$\Rightarrow 12^2 + y^2 = 13^2$$

$$y = 5 \text{ m}$$

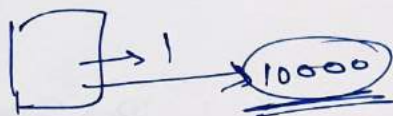
e.g. The total cost $C(x)$ in ~~₹~~ rupees, associated with the production of x units of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Find the ~~marginal~~ marginal cost when 3 units are produced.

(सीमांत लागत)

Ans.



Rate of change of total cost at a particular moment.

$$\text{Marginal cost} = \frac{d(C(x))}{dx}$$

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

$$\frac{d(C(x))}{dx} = (0.005)(3x^2) - 0.02(2x) + (30)(1) + 0$$

$(x=3)$

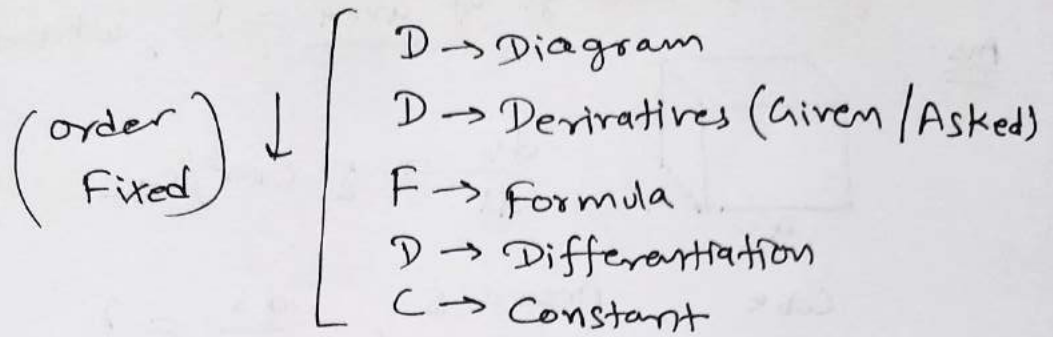
$$\frac{d(C(3))}{dx} = (0.005)(27) - (0.04)(6) + 30$$

$$= 0.135 - 0.24 + 30$$

$$= \underline{\underline{30.015 \text{ ₹}}}$$

$x \rightarrow$ units

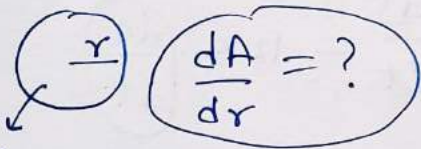
Exercise 6.1 Chapter 6 (Rate of change)



[Q.1] Rate of change of area of a circle with respect to radius r

- (i) when $r = 3\text{ cm}$ (ii) $r = 4\text{ cm}$.
(Constants)

Ans.



$$A = \pi r^2$$

diff. w.r.t. ' r '

$$\frac{dA}{dr} = \pi(2r)$$

(i) $r = 3$

$$\begin{aligned} \frac{dA}{dr} &= \pi(2 \times 3) \\ &= \underline{6\pi} \text{ cm}^2/\text{cm} \end{aligned}$$

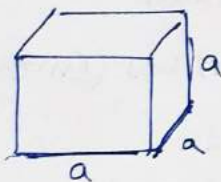
(ii) $r = 4\text{ cm}$

$$\begin{aligned} \frac{dA}{dr} &= \pi(2 \times 4) \\ &= 8\pi \text{ cm}^2/\text{cm} \end{aligned}$$

Q.2 Volume of a cube is increasing at $8 \text{ cm}^3/\text{s}$.

Rate of change of area = ? , when edge = 12 cm.
($a = 12 \text{ cm}$)

Ans.



Cube

$$\text{Area} = S, \quad \frac{dS}{dt} = ?$$

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{sec.}$$

Volume of a cube

$$V = a^3$$

diff. w.r.t 't'

$$\frac{dV}{dt} = 3a^2 \cdot \frac{da}{dt}$$

$$8 = 3(12)^2 \cdot \frac{da}{dt}$$

$$\Rightarrow \frac{8}{3(12)^2} = \frac{da}{dt}$$

Area of a cube (S)

$$S = 6a^2$$

diff. w.r.t. 't'

$$\frac{dS}{dt} = 12a \cdot \frac{da}{dt}$$

$$\frac{dS}{dt} = 12(12) \cdot \frac{8}{3(12)^2}$$

$$\frac{dS}{dt} = \frac{8}{3} \text{ cm}^2/\text{sec.}$$

Q.3 Radius increasing at $3 \text{ cm}/\text{s}$. = $\frac{dr}{dt}$
Rate of area change = ? = $\frac{dA}{dt}$ (when $r = 10 \text{ cm}$)

$$A = \pi r^2$$

by Diff. w.r.t to 't'

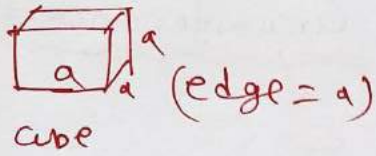
$$\frac{dA}{dt} = \pi (2r) \cdot \frac{dr}{dt}$$

$$= \pi (2 \times 10) \cdot (3) = \underline{\underline{60\pi \text{ cm}^2/\text{sec.}}}$$

Q.4 Edge of a variable cube is increasing at 3 cm/s.

Rate of change of volume = ? = $\frac{dV}{dt}$
(when edge = 10 cm)
 $a = 10$

$\frac{da}{dt}$



$$V = a^3$$

by diff. w.r. to 't'

$$\frac{dV}{dt} = 3a^2 \left(\frac{da}{dt} \right) = 3(10)^2 \cdot (3) = 900 \text{ cm}^3/\text{sec.}$$

Q.5 Waves moves in circles at the speed of 5 cm/s.

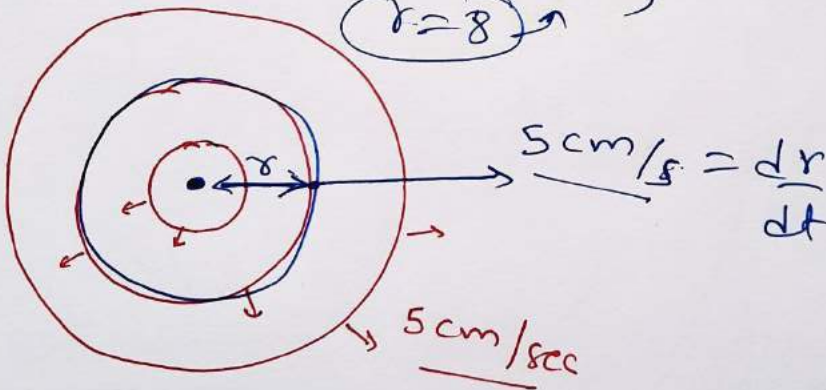
~~How~~ How fast the enclosed area increasing?
(when radius = 8 cm)

$$r = 8$$

$$\frac{dA}{dt}$$

$$\frac{dr}{dt}$$

Ans.



~~Area~~ Area = $A = \pi r^2$

$$\left(\frac{dA}{dt} \right) = \pi (2r) \cdot \frac{dr}{dt}$$

$$= \pi (2 \times 8) \cdot (5)$$

$$= \underline{80 \pi \text{ cm}^2/\text{sec.}}$$

Q.6 radius of a circle increasing

at the rate of $0.7 \text{ cm/s} = \frac{dr}{dt}$

Rate of change of ~~it~~ its Circumference = ?

$\frac{dc}{dt}$



$$C = 2\pi r$$

by diff. w.r.t. 't'

$$\begin{aligned} \frac{dc}{dt} &= 2\pi \frac{dr}{dt} = 2\pi(0.7) \\ &= \underline{1.4\pi \text{ cm/sec.}} \end{aligned}$$

Exercise 6.1 Chapter-6 (Roc)

Q.7

Length (x) decreasing

Rate = 5 cm/min.

$$\frac{dx}{dt} = -5$$

width (y) increasing

Rate = 4 cm/min.

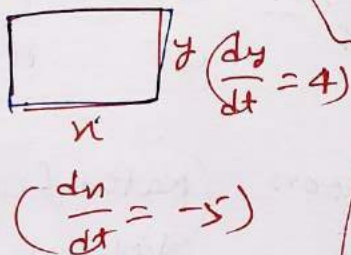
$$\frac{dy}{dt} = 4$$

when $x = 8$ cm, $y = 6$ cmConstants

D → Diagram
 D → Derivatives (Given/Asked)
 F → Formula
 D → Differentiation
 C → Constants
 (order)

(i) Rate of change of PerimeterPerimeter = P

$$\frac{dP}{dt} = ?$$



$$P = 2(x+y)$$

by diff. w.r.t. 't'

$$\frac{dP}{dt} = 2 \left(\frac{dx}{dt} + \frac{dy}{dt} \right)$$

$$= 2(-5 + 4)$$

$$= -2 \text{ cm/min.}$$

(ii) Rate of change ofArea ⇒ A

$$\frac{dA}{dt} = ?$$

$$A = x \cdot y$$

by diff. w.r.t. 't'

$$\frac{dA}{dt} = y \cdot \frac{dx}{dt} + x \cdot \frac{dy}{dt}$$

$$\frac{dA}{dt} = 6(-5) + 8(4)$$

$$= -30 + 32$$

$$= 2 \text{ cm}^2/\text{min.}$$

Q.8 Spherical Balloon, Inflated by pumping
 900 cubic cm of gas per second, Rate of
 change of radius = ?
 when $r = 15$ cm.

Ans.



$$\frac{dr}{dt} = ?$$

$$\left(\frac{dr}{dt}\right)$$

(Spherical)

$$\text{Volume} = V = \frac{4}{3} \pi r^3$$

by diff. w.r. to 't'

$$\frac{dV}{dt} = \frac{4}{3} \pi (3r^2) \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 900 \text{ cm}^3/\text{Sec.}$$

$$\Rightarrow 900 = 4\pi (15)^2 \cdot \left(\frac{dr}{dt}\right)$$

$$\Rightarrow \frac{900}{4\pi} = \frac{dr}{dt} = \frac{1}{\pi} \text{ cm/Sec.}$$

Q.9 Spherical Balloon, Rate of change of
Volume with radius = ?

when the later is 10 cm.

Ans.

$$\text{radius} = r = 10 \text{ cm.}$$

$$\frac{dV}{dr} = ?$$



volume of a sphere

$$V = \frac{4}{3} \pi r^3$$

by diff. w.r. to 'r'

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3} \pi (3r^2)$$

($r = 10$ cm)

$$\frac{dV}{dr} = 4\pi (10^2) = 400\pi \text{ cm}^3/\text{cm.}$$

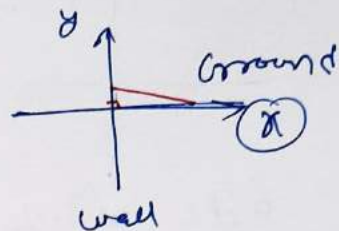
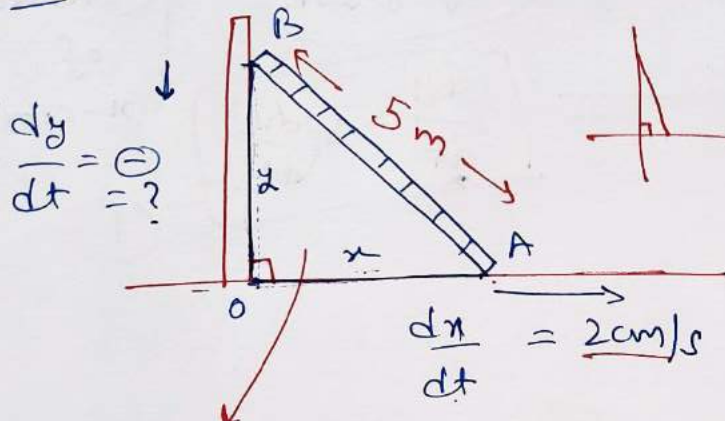
Q.10 Ladder 5 m long, leaning against a wall.

Bottom is pulled away from the wall at 2 cm/s.

How fast is its height decreasing = ?

when foot of ladder is 4m away from wall.

Ans:



Right angled Triangle,

By ~~Py~~ Pythagoras Theorem!

$$x^2 + y^2 = 5^2$$

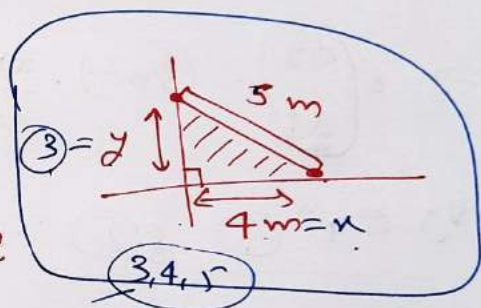
by diff. w.r. to 't'

$$(2x) \frac{dx}{dt} + (2y) \cdot \left(\frac{dy}{dt} \right) = 0$$

$$\Rightarrow y \cdot \left(\frac{dy}{dt} \right) = -x \cdot \frac{dx}{dt}$$

$$\Rightarrow \left(\frac{dy}{dt} \right) = -\frac{x}{y} \cdot \frac{dx}{dt} = -\frac{400}{300} \times 2 = \left(-\frac{8}{3} \text{ cm/s} \right)$$

Decreases at $\left(\frac{8}{3} \text{ cm/s} \right)$



$$x = 4\text{m} = 400\text{cm}$$

$$y = 3\text{m} = 300\text{cm}$$

$$\frac{dx}{dt} = 2\text{cm/s}$$

Q.11 Particle moves along $6y = x^3 + 2$.

Find points on the curve at which y -coordinate is changing 8 times as fast as the x -coordinate.

Ans.



Rate of change of 'y-coor.' = 8 (Rate of change of 'x-coor.')

$$\frac{dy}{dt} = 8 \left(\frac{dx}{dt} \right)$$

①

$$6y = x^3 + 2$$

by diff. w.r. to 't'

$$\Rightarrow 6 \left(\frac{dy}{dt} \right) = (3x^2) \cdot \frac{dx}{dt} \quad \text{--- ②}$$

by eqn ① & ② $\Rightarrow \cancel{6} \left(8 \frac{dx}{dt} \right) = \cancel{3} x^2 \cdot \cancel{\frac{dx}{dt}}$

$$\Rightarrow 16 = x^2$$

$$x = \pm 4$$

$$x = 4$$

$$x = -4$$

$$6y = x^3 + 2$$

$$6y = x^3 + 2$$

$$\Rightarrow 6y = 4^3 + 2$$

$$\Rightarrow 6y = -64 + 2$$

$$\Rightarrow 6y = 66$$

$$\Rightarrow \frac{6y}{3} = \frac{-62}{3}$$

$$y = 11$$

$$y = -\frac{31}{3}$$

$$(4, 11)$$

$$\left(-4, -\frac{31}{3} \right)$$


Exercise 6.1

Chapter 6

(Rate of change)

Q.12 Radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when radius is 1 cm?

Ans.



$$\frac{dr}{dt} = \frac{1}{2} \text{ cm/s} \quad \frac{dv}{dt} = ?$$

$$V = \frac{4}{3} \pi r^3$$

by differentiating w.r. to 't'

$$\frac{dv}{dt} = \frac{4}{3} \pi (3r^2) \cdot \frac{dr}{dt}$$

$$r = 1$$

$$\frac{dr}{dt} = \frac{1}{2}$$

$$\Rightarrow \frac{dv}{dt} = \frac{4}{3} \pi \cdot 3 \cdot 1 \cdot \frac{1}{2} = 2\pi \text{ cm}^3/\text{Sec.}$$

Q.13 A balloon (always spherical), has a variable diameter $\frac{3}{2}(2x+1)$, Find rate of change of its volume with respect to x .

Ans.

$$\text{Dia.} = \frac{3}{2}(2x+1)$$

$$r = \frac{3}{4}(2x+1)$$



$$V = \frac{4}{3} \pi r^3$$

$$V = \frac{4}{3} \pi \left(\frac{3}{4} (2x+1) \right)^3$$

$$V = \frac{4}{3} \pi \left(\frac{3}{4} \right)^3 (2x+1)^3 = \left(\frac{3}{4} \right)^2 \pi (2x+1)^3 \rightarrow$$

$$V = \left(\frac{3}{4}\right)^2 \cdot \frac{\pi}{x} \cdot \underbrace{(2x+1)^3}$$

$$\frac{d(x^3)}{dx} = 3x^2$$

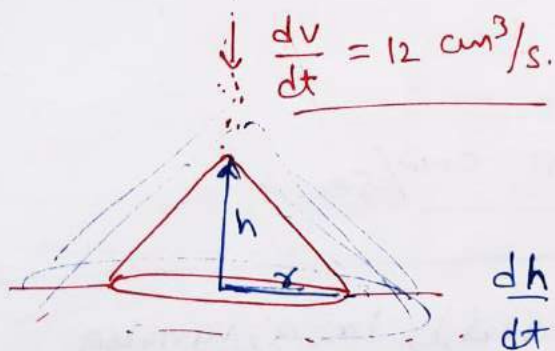
by diff. w.r.t. 'x'

$$\frac{dV}{dx} = \frac{9}{8} \cdot \pi \cdot 3(2x+1)^2 \cdot \cancel{(2+0)}$$

$$= \frac{27}{8} \pi (2x+1)^2$$

Q.14 Sand is pouring at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

(Constant)



$$h = \frac{1}{6} \text{ of } r$$

$$h = \frac{r}{6} \text{ (given)}$$

$$6h = r$$

$$\text{Volume of cone} = V = \frac{1}{3} \pi r^2 h$$

$$\Rightarrow V = \frac{1}{3} \pi (6h)^2 h$$

$$\Rightarrow \boxed{V = 12\pi h^3}$$

by diff. w.r.t. 't'

$$\Rightarrow \frac{dV}{dt} = 12\pi (3h^2) \cdot \left(\frac{dh}{dt}\right)$$

12

$h = 4$

$$12 = 12\pi (3 \cdot 4^2) \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{1}{48\pi} = \frac{dh}{dt}$$

cm/sec.

Q.15 Total cost $C(x)$ in Rupees

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Marginal cost = ? when 17 units are produced.

→ Rate of change of total cost

Ans.

$$\text{Marginal cost} = \frac{d(\text{Total cost})}{dx} = \frac{d(C(x))}{dx}$$

$$= \frac{d(0.007x^3 - 0.003x^2 + 15x + 4000)}{dx}$$

$$= 0.007(3x^2) - 0.003(2x) + 15$$

$$= \text{(when } x=17)$$

$$= 0.007(3 \times 289) - 0.003(34) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 21.069 - 0.102 = \underline{\underline{20.967}}$$

Q.16 Total Revenue $R(x)$ sale x units

Find marginal revenue when $x=7$

Ans. $R(x) = 13x^2 + 26x + 15$

$$\text{Marginal Revenue} = \frac{d(R(x))}{dx}$$

$$= \frac{d(13x^2 + 26x + 15)}{dx}$$

$$= 13(2x) + 26$$

$x=7$ put

$$MR = 13(2 \times 7) + 26$$

$$= 13 \times 14 + 26$$

$$= 182 + 26$$

$$= \underline{\underline{208 \text{ ₹}}}$$

Q.17 Rate of change of the area of a circle with respect its radius r at $r=6$ cm is —

- (A) 10π (B) 12π (C) 8π (D) 11π .

Ans. Area of a circle $= A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = \pi(2r)$$

$$(r=6 \text{ cm.})$$

$$\frac{dA}{dr} = \pi(2 \times 6) = 12\pi$$

Q.18 The total revenue in rupees ~~received~~ received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x=15$ is —

- (A) 116 (B) 96 (C) 90 (D) 126

$$\text{Marginal Revenue} = \frac{d(R(x))}{dx}$$

$$= \frac{d(3x^2 + 36x + 5)}{dx} = 3(2x) + 36$$

when $x=15$

$$\text{M.R.} = 6(15) + 36 = 90 + 36 = 126 \text{ ₹}$$

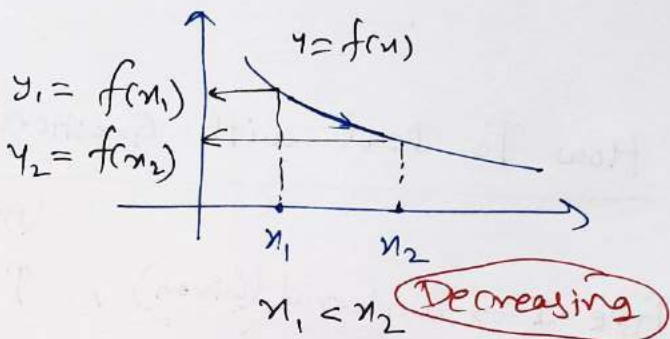
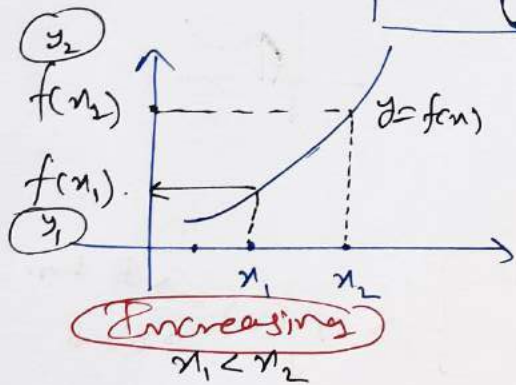
Increasing and Decreasing Functions.

(वर्धमान)

(ह्रासमान)



Always move from left to Right
 $x \uparrow$



$f(x_1) < f(x_2)$

$f(x_1) > f(x_2)$

$\frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$
 $\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{+}{+} = +$

$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$
 $= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-}{+}$

Slope = +

= -

$f'(x) > 0$ ★

$f'(x) < 0$

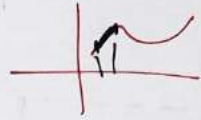
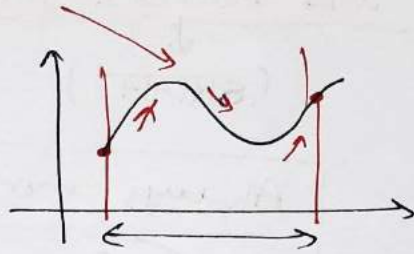
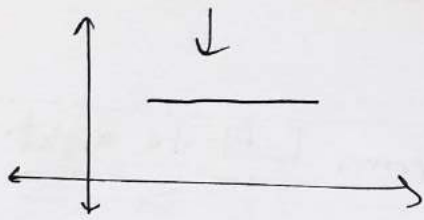
Increasing
 $f'(x) \geq 0$

Decreasing
 $f'(x) \leq 0$

Strictly Increasing
 $f'(x) > 0$

Strictly Decreasing
 $f'(x) < 0$

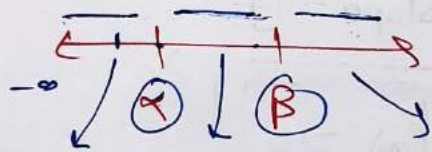
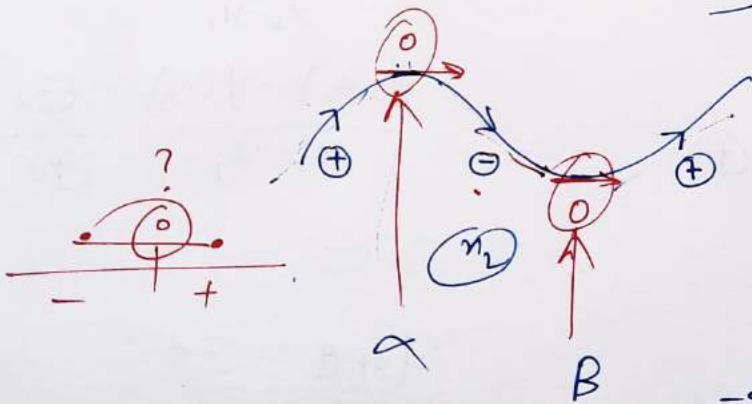
Neither Increasing Nor Decreasing



How To Deal with Questions?

Type-I \rightarrow Interval (Given), \uparrow / \downarrow (Asked) $\rightarrow f'(x)$ $\rightarrow \oplus \uparrow$ En. $\rightarrow \ominus \downarrow$
 (Sign check)

Type-II \rightarrow Interval (Asked), \uparrow / \downarrow (Given) $\rightarrow f'(x) = 0$
 \downarrow Roots $x = ?$
 \downarrow (A, B)
Numberline



Let check (x_1) check (x_2) check (x_3)

$f'(x_1)$
 $(+)$

$f'(x_2)$
 $(-)$

$f'(x_3)$
 $(+)$

Increasing

Decreasing

$(+)$

e.g. show that $f(x) = 3x^3 + 7$ is increasing
on \mathbb{R} .

Ans.

$$f(x) = 3x^3 + 7$$

$$f'(x) = 9x^2 \quad x \in \mathbb{R}$$

$$()^2 \geq 0$$

$$x^2 \geq 0$$

$$9x^2 \geq 0$$

$f'(x) \geq 0 \Rightarrow \therefore f(x)$ is always increasing
on \mathbb{R} .

Slope = (+)

e.g. Find the interval in which

$f(x) = 4x^3 - 6x^2 - 72x + 30$ is (I) Increasing.

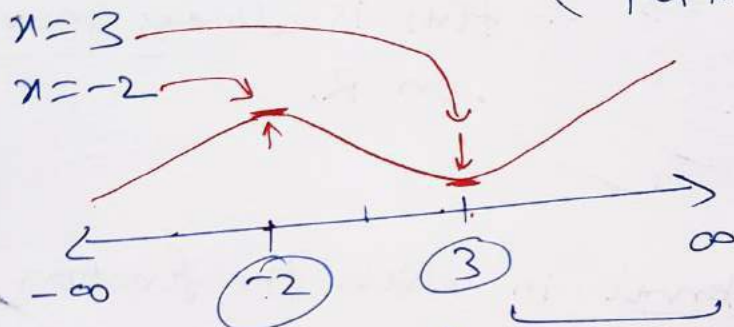
(II) Decreasing.

Ans. $f'(x) = 12x^2 - 12x - 72$

$$\Rightarrow f'(x) = 12(x^2 - x - 6)$$

$$\Rightarrow f'(x) = 12(x^2 - 3x + 2x - 6)$$

$$\Rightarrow f'(x) = 12(x-3)(x+2) = 0 \quad (\text{For turning points})$$



$f'(x) = 12(x-3)(x+2)$	$(-)(-)$	$(-)(+)$	$(+)(+)$
	\oplus	\ominus	\oplus

Increasing Decreasing Increasing

(I) Increasing Interval $\rightarrow (-\infty, -2) \cup (3, \infty)$

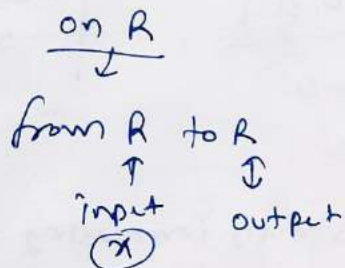
(II) Decreasing Interval $\rightarrow (-2, 3)$

Exercise - 6.2 Chapter - 6

(Increasing and Decreasing Functions)

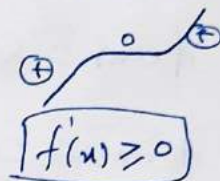
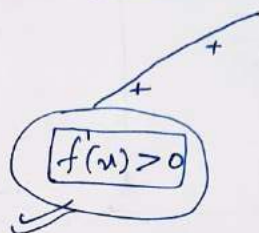
Q.1 Show that the function given by $f(x) = 3x + 17$ is strictly increasing on \mathbb{R} .

Solution: $f(x) = 3x + 17$



Strictly Inc.

Inc.



$x \in \mathbb{R}$

$$\boxed{f'(x) = 3} = \oplus \text{ve}$$

$$\boxed{f'(x) > 0}, \quad \underline{x \in \mathbb{R}}$$

$\therefore f(x) = 3x + 17$ is always strictly increasing on $x \in \mathbb{R}$.

Q.2 Show that the function given by $f(x) = e^{2x}$ is strictly increasing on \mathbb{R} .

Solⁿ:

$$f(x) = e^{2x}$$

$e = \text{Euler's no.} = \underline{2.718...}$

$$\Rightarrow f'(x) = e^{2x} \cdot 2$$

(+)^(Power)

$$(+)^+ = \oplus \checkmark$$

$$\boxed{f'(x) = \underbrace{2}_{+} \cdot \underbrace{e^{2x}}_{+}} = \oplus \text{ve.}$$

$$(+)^{-} = \oplus \checkmark$$

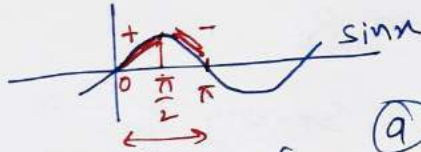
$$f'(x) = 2e^{2x} > 0, \quad x \in \mathbb{R}$$

$\therefore f(x)$ is strictly increasing on $x \in \mathbb{R}$.

Q.3 Show that the function given by $f(x) = \sin x$ is -

- (a) strictly increasing in $(0, \frac{\pi}{2})$ (b) strictly decreasing in $(\frac{\pi}{2}, \pi)$
 (c) neither increasing nor decreasing in $(0, \pi)$

Solution.



$$f(x) = \sin x$$

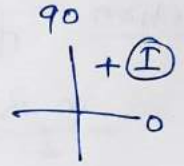
$$f'(x) = \underline{\cos x}$$

(a) $I_n (0, \frac{\pi}{2})$

$$\cos x > 0$$

$$f'(x) > 0$$

$\therefore f(x)$ is strictly increasing in $(0, \frac{\pi}{2})$.

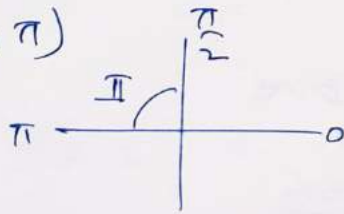


(b) $I_n (\frac{\pi}{2}, \pi)$

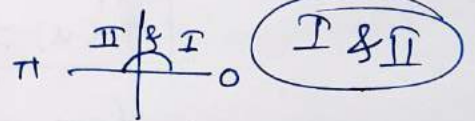
✓

$$\cos x < 0$$

$f'(x) < 0 \therefore$ strictly decreasing.



(c) $I_n (0, \pi)$



$$\cos x > 0 \quad \textcircled{I}$$

$$\cos x < 0 \quad \textcircled{II}$$

$$\Rightarrow \text{not } f'(x) > 0$$

$$\text{not } f'(x) < 0$$

\therefore neither increasing nor decreasing in $(0, \pi)$

- Q.4** Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is ~~strictly increasing~~ $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$
- (a) strictly increasing (b) strictly decreasing.

Ans.

$$f(x) = 2x^2 - 3x$$

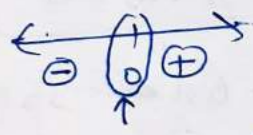
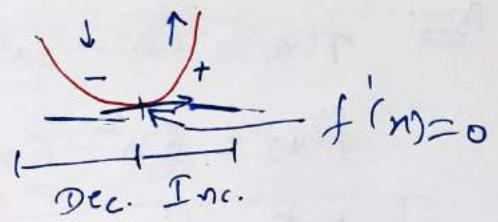
Diff.

$$f'(x) = 4x - 3$$

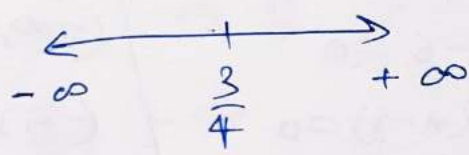
Put $f'(x) = 0$

$$\Rightarrow 4x - 3 = 0$$

$$\Rightarrow \boxed{x = \frac{3}{4}}$$



~~strictly increasing~~



$$f'(x) = 4x - 3 \quad \text{for } \left(-\infty, \frac{3}{4}\right) \rightarrow f'(x) < 0$$

(Strictly decreasing)

$$\text{for } \left(\frac{3}{4}, \infty\right) \rightarrow f'(x) > 0$$

Strictly Increasing.

Q.5 Find the Intervals in which the function

$$f(x) = 2x^3 - 3x^2 - 36x + 7 \text{ is}$$

(a) strictly increasing.

(b) strictly decreasing.

Ans. $f(x) = 2x^3 - 3x^2 - 36x + 7$

$$f'(x) = 6x^2 - 6x - 36$$

$$= 6(x-3)(x+2)$$

Put $f'(x) = 0$

$$\Rightarrow 6x^2 - 6x - 36 = 0$$

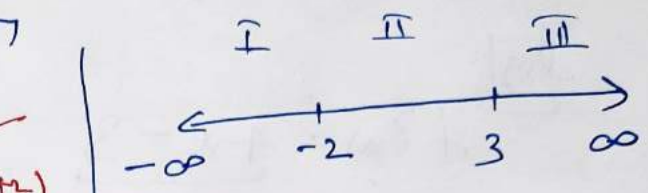
$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x^2 - 3x + 2x - 6 = 0$$

$$\Rightarrow x(x-3) + 2(x-3) = 0$$

$$\Rightarrow (x-3)(x+2) = 0$$

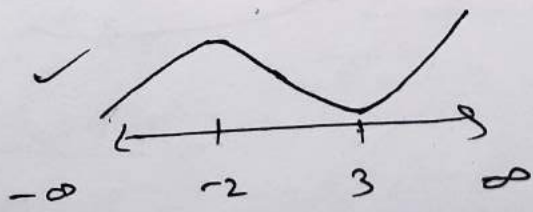
$$x = 3, -2$$



Interval	$f'(x)$ Sign	\uparrow/\downarrow
$(-\infty, -2)$	$(-)(-) = +$	\uparrow
$(-2, 3)$	$(-)(+) = -$	\downarrow
$(3, \infty)$	$(+)(+) = +$	\uparrow

(a) Strictly increasing, $(-\infty, -2) \cup (3, \infty)$

(b) Strictly decreasing, $(-2, 3)$



Q.6 Find the intervals for strictly increasing or decreasing

a) $x^2 + 2x - 5$

$f(x) = x^2 + 2x - 5$

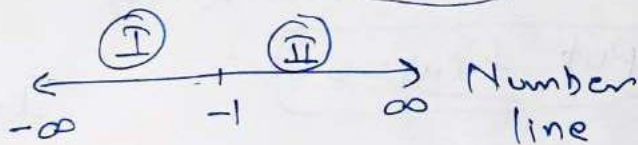
$f'(x) = 2x + 2 = 2(x+1)$

Put $f'(x) = 0$

$\Rightarrow 2x + 2 = 0$

$\Rightarrow x + 1 = 0$

$x = -1$



Interval	Sign of $f'(x)$	\uparrow/\downarrow
I $(-\infty, -1)$	(-)	\downarrow (Dec.)
II $(-1, \infty)$	(+)	\uparrow (Inc.)

Strictly Increasing in $(-1, \infty) = x > -1$

Strictly Decreasing in $(-\infty, -1) = x < -1$

b) $f(x) = 10 - 6x - 2x^2$

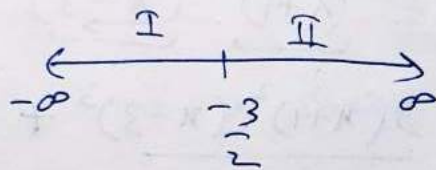
$f'(x) = -6 - 4x$

$f'(x) = -2(3 + 2x)$

Put $f'(x) = 0$

$\Rightarrow -2(3 + 2x) = 0$

$x = -\frac{3}{2}$



Intervals	Sign of $f'(x)$	\uparrow/\downarrow
I $(-\infty, -\frac{3}{2})$	(+)	\uparrow (Increase)
II $(-\frac{3}{2}, \infty)$	(-)	\downarrow (Decrease)

St. Increasing

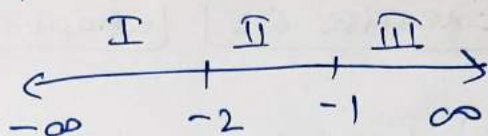
St. Decreasing

$$\textcircled{c} f(x) = -2x^3 - 9x^2 - 12x + 1$$

$$\Rightarrow f'(x) = -6x^2 - 18x - 12$$

$$\Rightarrow f'(x) = -6(x^2 + 3x + 2)$$

$$\Rightarrow f'(x) = -6(x+1)(x+2)$$



Put $f'(x) = 0$

$$-6(x+1)(x+2) = 0$$

$$x = -1, -2$$

Interval	Sign of $f'(x)$	\uparrow / \downarrow
$\textcircled{\text{I}} (-\infty, -2)$	$(-)$	\downarrow (Dec.)
$\textcircled{\text{II}} (-2, -1)$	$(+)$	\uparrow (Inc.)
$\textcircled{\text{III}} (-1, \infty)$	$(-)$	\downarrow (Dec.)

strictly increasing in $(-2, -1)$

strictly decreasing in $(-\infty, -2) \cup (-1, \infty)$

$$\textcircled{d} f(x) = 6 - 9x - x^2$$

$$\textcircled{e} f(x) = (x+1)^3 \cdot (x-3)^3 \quad \boxed{(u \cdot v)' = u' \cdot v + u \cdot v'}$$

$$f'(x) = 3(x+1)^2 \cdot (x-3)^3 + (x+1)^3 \cdot 3(x-3)^2$$

$$f'(x) = 3(x+1)^2(x-3)^2(x-3 + x+1)$$

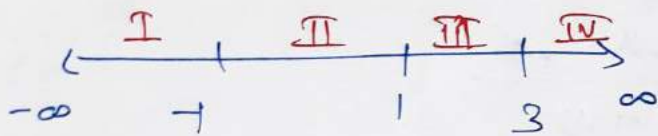
$$f'(x) = 3(x+1)^2(x-3)^2(2x-2)$$

$$f'(x) = 6(x+1)^2(x-3)^2(x-1)$$

Now put $f'(x) = 0$

$$\Rightarrow 6(x+1)^2(x-3)^2(x-1) = 0$$

$$\underline{x = -1}, \underline{x = 3}, \underline{x = 1}$$



$$f'(x) = 6(x+1)^2(x-3)^2(x-1)$$

\downarrow \downarrow \downarrow \downarrow
 \oplus \oplus \oplus \odot

Intervals	Sign of $f'(x)$	\uparrow/\downarrow
$(-\infty, -1)$	$(-)$	\downarrow (Decreasing)
$(-1, 1)$	$(-)$	\downarrow Decreasing
$(1, 3)$	$(+)$	\uparrow (Inc.) ✓
$(3, \infty)$	$(+)$	\uparrow Increasing. ✓

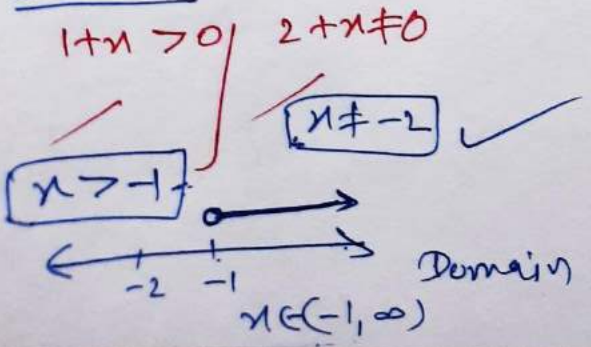
Strictly increasing in $(1, 3) \cup (3, \infty)$

Strictly decreasing in $(-\infty, -1) \cup (-1, 1)$

Q.7 Show that $y = \log(1+x) - \frac{2x}{2+x}$, $(x > -1)$ is an increasing function of x throughout its Domain.

Ans. $f(x) = \log(1+x) - \frac{2x}{2+x}$

Domain



Domain $(-1, \infty)$

$$f(x) = \log(1+x) - \frac{2x}{(2+x)} \quad \left(\frac{u}{v} \right)$$

$x > -1$
Domain $(-1, \infty)$

$$f'(x) = \frac{1}{1+x} - \left[\frac{(2)(2+x) - 2x(1)}{(2+x)^2} \right] \rightarrow \left(\frac{u}{v} \right)' = \frac{u'v - u \cdot v'}{v^2}$$

$$f'(x) = \frac{1}{1+x} - \left[\frac{4 + 2x - 2x}{(2+x)^2} \right]$$

$$f'(x) = \frac{(2+x)^2 - (1+x) \cdot 4}{(1+x) \cdot (2+x)^2}$$

$$f'(x) = \frac{\cancel{4} + 4x + x^2 - \cancel{4} - 4x}{(1+x)(2+x)^2}$$

$$f'(x) = \frac{x^2 \rightarrow (+)}{(1+x) \cdot (2+x)^2 \rightarrow (+)} \geq 0$$

$\hookrightarrow (+)$

Domain

$$\left[\begin{array}{l} x \in (-1, \infty) \\ x > -1 \\ \frac{1+x}{1} > 0 \\ (+) \end{array} \right]$$

$$f'(x) = (+)ve$$

$$\boxed{f'(x) \geq 0}$$

$f(x) \rightarrow$ increasing in its domain,

Exercise-6.2

chapter-6

↑ & ↓

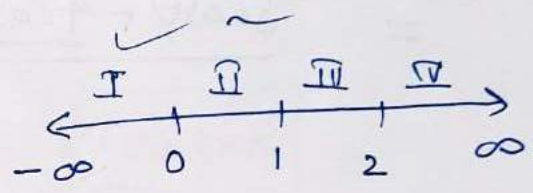
Q.8 Find the value of x for which $y = [x(x-2)]^2$ is an increasing function.

Ans. $f(x) = [x(x-2)]^2 = [x^2 - 2x]^2$

$f'(x) = 2(x^2 - 2x) \cdot (2x - 2)$

$f'(x) = 4(x)(x-2)(x-1)$

Put $f'(x) = 0$



$\Rightarrow 4(x)(x-2)(x-1) = 0$

$x = 0, 2, 1$

Interval	Sign of $f'(x)$	↑/↓
<u>I</u> $(-\infty, 0)$	$(-)$	Decreasing ↓
<u>II</u> $(0, 1)$ ✓	$(+)$	↑ <u>Increasing</u>
<u>III</u> $(1, 2)$	$(-)$	↓ Decreasing
<u>IV</u> $(2, \infty)$ ✓	$(+)$	↑ <u>Increasing</u>

Increasing function in $(0, 1) \cup (2, \infty)$

Q.9 Prove that $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ is an increasing function of θ in $[0, \frac{\pi}{2}]$.

Ans.

$$y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - u.v'}{v^2}$$

$$\frac{dy}{d\theta} = \frac{[4 \cos \theta] \cdot (2 + \cos \theta) - 4 \sin \theta \cdot [-\sin \theta]}{(2 + \cos \theta)^2} - \frac{1}{1}$$

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta - (2 + \cos \theta)^2}{(2 + \cos \theta)^2}$$

$$= \frac{8 \cos \theta + \cancel{4(\cos^2 \theta + \sin^2 \theta)} - \cancel{4} - \cos^2 \theta - \cancel{4 \cos \theta}}{(2 + \cos \theta)^2}$$

$$\frac{dy}{d\theta} = \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2}$$

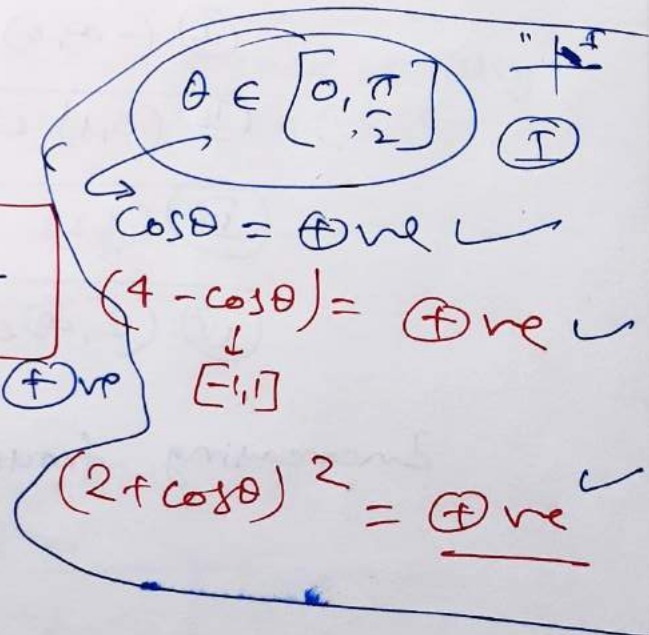
$$\frac{dy}{d\theta} = \frac{\cos \theta \cdot (4 - \cos \theta)}{(2 + \cos \theta)^2}$$

= (+)ve

$\therefore \frac{dy}{d\theta} > 0$ for $\theta \in [0, \frac{\pi}{2}]$

$$\therefore y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$$

is an increasing function in $[0, \frac{\pi}{2}]$



Q. 10 Prove that logarithmic function is strictly increasing on $(0, \infty)$.

Solution

$$f(x) = \log x$$

$$\log_e(x) = \ln(x)$$

Natural log.

Domain: $\log(x)$
 $x > 0$
 $x \in (0, \infty)$

$$f'(x) = \frac{1}{x}$$

$$x = \oplus \text{ve}$$

Here $x > 0$

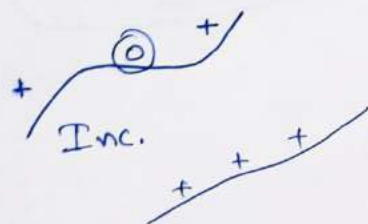
$$\Rightarrow 0 < x < \infty$$

$$\Rightarrow \infty > \frac{1}{x} > 0$$

(by Reproccating)

$$\frac{1}{x} > 0$$

$$\Rightarrow f'(x) > 0$$



Strictly Inc

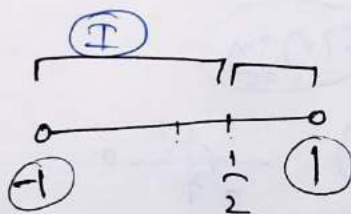
$\therefore \log x$ is strictly increasing in $(0, \infty)$

Q. 11 Prove that $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

Ans: $f(x) = x^2 - x + 1$

$$f'(x) = 2x - 1$$

Put $f'(x) = 0$



$$\Rightarrow 2x - 1 = 0$$

$$\Rightarrow x = \frac{1}{2}$$

Interval	Sign of $f'(x)$	\uparrow/\downarrow
$(-1, \frac{1}{2})$	$(-)$	\downarrow
$(\frac{1}{2}, 1)$	(\oplus)	\uparrow

$\therefore f'(x) > 0, \forall x \in (\frac{1}{2}, 1)$ & $f'(x) < 0, \forall x \in (-1, \frac{1}{2})$

$\therefore f(x)$ is ~~neither~~ neither $(-1, 1)$

Q. 12) which of the following functions are strictly decreasing on $(0, \frac{\pi}{2})$? $x \in (0, \frac{\pi}{2})$

- (A) ~~cos x~~ (B) ~~cos 2x~~ (C) ~~cos 3x~~ (D) ~~tan x~~

Ans
 (A) $f(x) = \cos x$
 $f'(x) = \frac{-\sin x}{(-)(+)}$
 $f'(x) = \ominus ve$
 $f(x)$ strictly decreasing.

(B) $f(x) = \cos 2x$
 $f'(x) = -2 \sin(2x)$
 $x \in (0, \frac{\pi}{2})$
 $2x \in (0, \pi)$
 $\pi \xrightarrow{\frac{\pi}{2} \quad \frac{3\pi}{2}} 0$
 $\sin 2x \in (0, 1)$
 $\sin 2x > 0$
 $\frac{-2 \sin 2x}{\ominus \oplus} < 0 \quad \therefore f'(x) < 0$
Strictly Dec.

(C) $\cos 3x$
 $f(x) =$

$f'(x) = -3 \sin 3x$
 $x \in (0, \frac{\pi}{2}) \quad \frac{+}{-} \frac{+}{+} 0$
 $3x \in (0, \frac{3\pi}{2}) \quad \frac{3\pi}{2}$

$\sin 3x > 0$ for some values of x

$\sin 3x < 0$ for some values of x

(D) $f(x) = \tan x$
 $f'(x) = \sec^2 x$
 $= (\sec x)^2 > 0$

Increasing

अतः $f'(x) > 0$, अतः $f'(x) < 0$

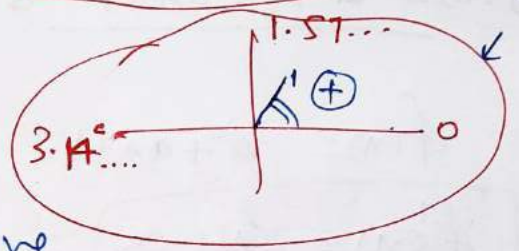
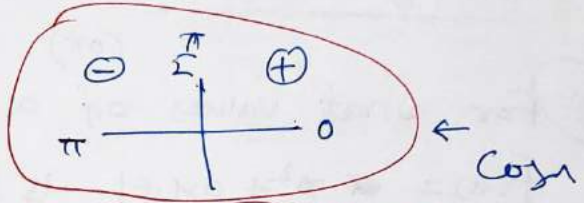
Q.13 On which of the following intervals $f(x) = x^{100} + \sin x - 1$ is strictly decreasing?

- (A) $(0, 1)$
 (B) $(\frac{\pi}{2}, \pi)$
 (C) $(0, \frac{\pi}{2})$
 (D) None of these

$f'(x) < 0$

Ans $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100 \cdot x^{99} + \cos x$



Option (A) $x \in (0, 1)$

$f'(x) = \underbrace{100 \cdot x^{99}}_{\oplus} + \underbrace{\cos x}_{\oplus} = \oplus ve$

$f'(x) > 0$ (increasing)

Option (B) $x \in (\frac{\pi}{2}, \pi)$ $x \in (1.57\dots, 3.14\dots)$

$f'(x) = 100 \cdot x^{99} + \cos x$

$(\frac{\pi}{2}, \pi)$
 ↑↑↑
 \ominus
 = $\oplus ve$
 $f'(x) > 0$
 $\oplus ve$
 no.
 \ominus
 Increasing

Option (C) $x \in (0, \frac{\pi}{2})$

$f'(x) = 100 \cdot x^{99} + \cos x = \oplus ve$

\oplus \oplus (Increasing)

Exercise 6.2 chapter 6

Q.14 Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$. (pdf)

Book For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?

Ans.
 $f(x) = x^2 + ax + 1$

$f'(x) > 0$

$f'(x) = 2x + a > 0$
 $x \in (1, 2)$

$1 < x < 2$

$\Rightarrow 2 < 2x < 4$

$\Rightarrow 2 + a < (2x + a) < 4 + a$

$\Rightarrow (2 + a) < f'(x) < 4 + a$

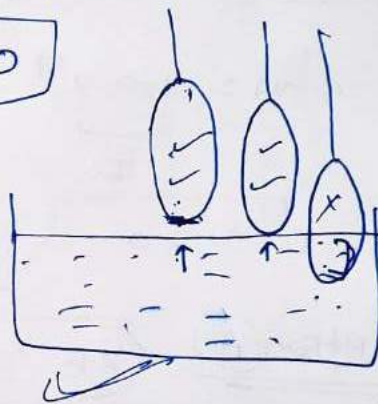
$\therefore f'(x) > 0$

\therefore minimum value of $f'(x) \geq 0$

$\Rightarrow 2 + a \geq 0$

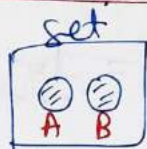
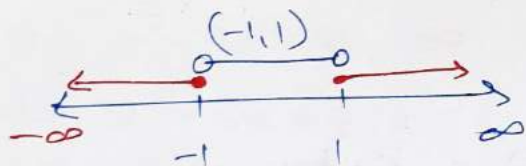
$\Rightarrow a \geq -2$

least value of $a = -2$



Q.15 Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I .

Ans.



$A \cap B = \phi$

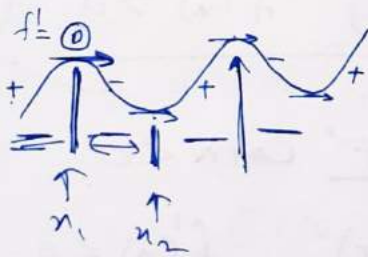
Disjoint

$I \in (-\infty, -1] \cup [1, \infty)$

$f(x) = x + \frac{1}{x}$

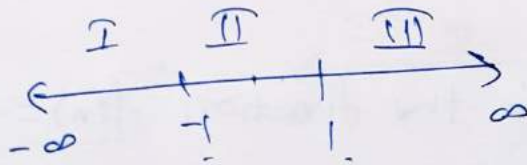
$f'(x) = 1 - \frac{1}{x^2} = x^2 - \left(\frac{1}{x}\right)^2 = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

$f'(x) = 0$ put



$\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) = 0$

$x = -1$ $x = 1$



$f'(x) = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

Interval,	Sign of $f'(x)$	\uparrow / \downarrow
<u>I</u> $(-\infty, -1] :$	\oplus	<u>\uparrow (Inc.)</u>
<u>II</u> $(-1, 1)$	\ominus	<u>\downarrow (Dec.)</u>
<u>III</u> $[1, \infty) :$	\oplus	<u>\uparrow (Inc.)</u>

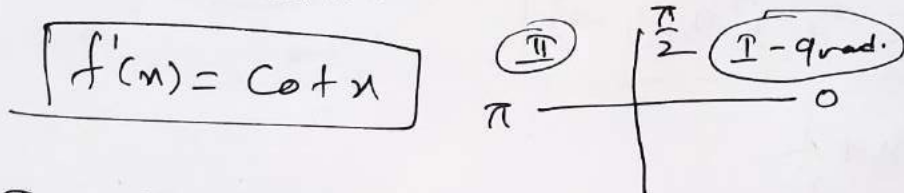
$\therefore f(x) = x + \frac{1}{x}$ is strictly increasing in $(-\infty, -1) \cup (1, \infty)$

I

Q.16 Prove that the function $f(x) = \log \sin x$ is strictly increasing on $(0, \frac{\pi}{2})$ and strictly decreasing on $(\frac{\pi}{2}, \pi)$.

Ans. $f(x) = \log \sin x$

$$f'(x) = \frac{1}{\sin x} \cdot (\cos x) = \cot x$$



In I-quadrant $\cot x > 0$

$x \in (0, \frac{\pi}{2})$ $f'(x) > 0$ (Strictly Inc.)

In II-quadrant $\cot x < 0$

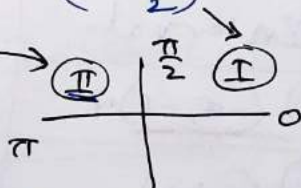
$x \in (\frac{\pi}{2}, \pi)$, $f'(x) < 0$ (Strictly Decreasing)

Q.17 Prove that the function $f(x) = \log \cos x$ is strictly decreasing on $(0, \frac{\pi}{2})$ and strictly increasing on $(\frac{\pi}{2}, \pi)$.

Ans. $f(x) = \log \cos x$

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x)$$

$f'(x) = -\tan x$



I-quadrant $x \in (0, \frac{\pi}{2})$

$\tan x > 0$
 $-\tan x < 0 \therefore f'(x) < 0$
 Strictly Decreasing

II-quadrant $x \in (\frac{\pi}{2}, \pi)$, $\tan x < 0$
 $-\tan x > 0 \therefore f'(x) > 0$ Strictly Inc.

Q. 18 Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in \mathbb{R} .

Ans. $x \in \mathbb{R}$.

$$f(x) = x^3 - 3x^2 + 3x - 100$$

$$f'(x) = 3x^2 - 6x + 3$$

$$f'(x) = 3(x^2 - 2x + 1)$$

$$f'(x) = 3(x-1)^2$$

For $x \in \mathbb{R}$

$$3(x-1)^2 \geq 0$$

$$f'(x) \geq 0$$

$f(x)$ is increasing for $x \in \mathbb{R}$.

$$(x-1)^2$$

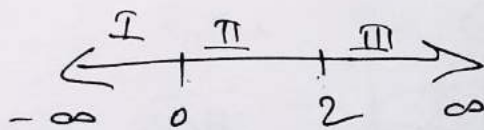
Perfect Square ≥ 0

Q. 19 The interval in which $y = x^2 \cdot e^{-x}$ is increasing is —

- (A) $(-\infty, \infty)$ (B) $(2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

Ans. $y = x^2 \cdot e^{-x}$

$$f'(x) = 2x \cdot e^{-x} - x^2 \cdot e^{-x}$$



$$f'(x) = x(2-x) \cdot e^{-x}$$

Put $f'(x) = 0$

$$x(2-x) \cdot e^{-x} = 0$$

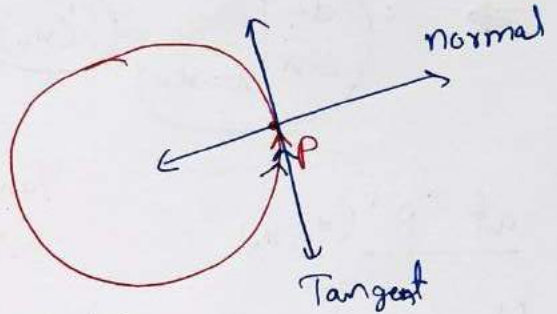
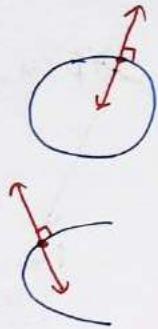
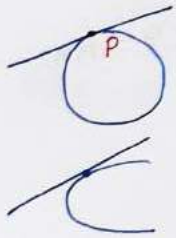
$x=0$ $x=2$ $e^{-x} \neq 0$

Interval	Sign of $f'(x)$	\uparrow/\downarrow
(I) $(-\infty, 0)$	$-$	\downarrow (Dec.)
(II) $(0, 2)$	$+$	\uparrow (Inc.)
(III) $(2, \infty)$	$-$	\downarrow (Dec.)

Tangents and Normals

[स्पर्श रेखा]

[अमिलता]



At point 'P'

Graph \parallel Tangent

Normal \perp Tangent

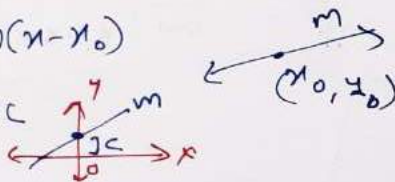
Normal \perp Graph

Revision:

\hookrightarrow Line

$$(y - y_0) = m(x - x_0)$$

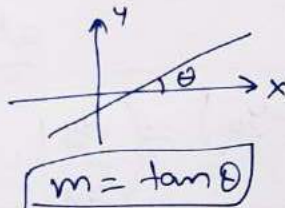
$$y = mx + c$$



Slope (m)

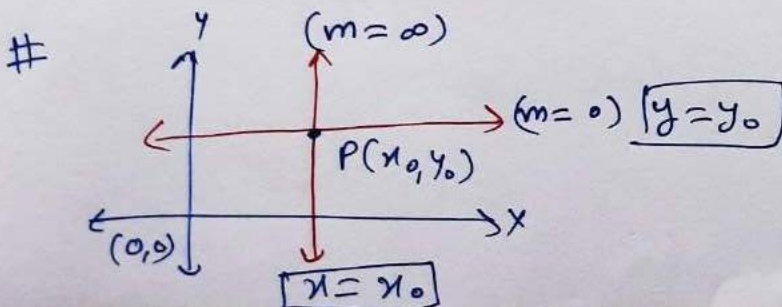
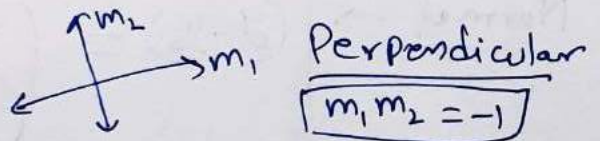
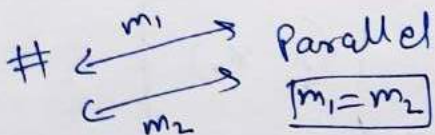
(x_1, y_1) (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$$m = \frac{dy}{dx}$$

 Diff.



Equations of Tangents and Normals :→

Slope of Curve $y = f(x)$
at $x = x_0$

$$\Rightarrow = \left. \frac{dy}{dx} \right|_{x=x_0} = f'(x_0)$$

at 'P' (x_0, y_0)

Normal \perp Tangent \parallel Curve
 (m_2) (m_1) $\left(\left. \frac{dy}{dx} \right|_{x=x_0} \right)$

$$m_1 m_2 = -1$$

$$m_2 = -\frac{1}{m_1}$$

$$m_1 = \left. \frac{dy}{dx} \right|_{x=x_0}$$

$$m_2 = -\frac{1}{\left. \frac{dy}{dx} \right|_{x=x_0}}$$

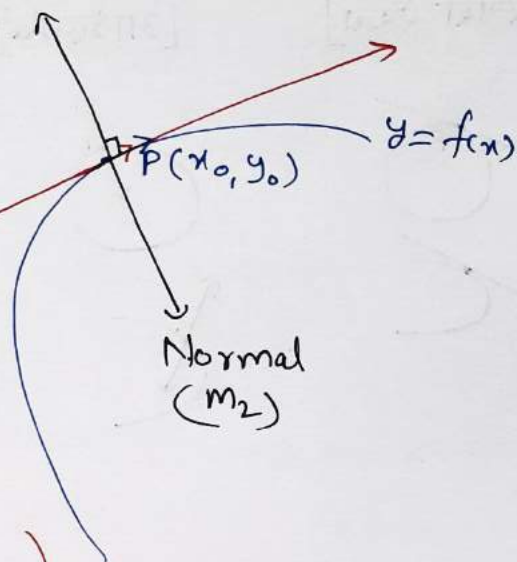
at Point $P(x_0, y_0)$

Tangent →

$$(y - y_0) = \left(\left. \frac{dy}{dx} \right|_{x=x_0} \right) \cdot (x - x_0)$$

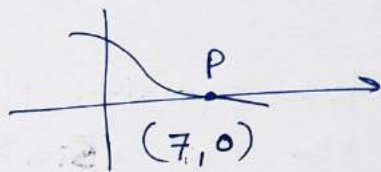
Normal →

$$(y - y_0) = \left(-\frac{1}{\left. \frac{dy}{dx} \right|_{x=x_0}} \right) (x - x_0)$$



e.g Find the slope of the tangent and normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the Point where it cuts the x-axis. Hence find equations of tangent and normal.

Ans. $y = \frac{(x-7)}{(x-2)(x-3)}$



For the point P on

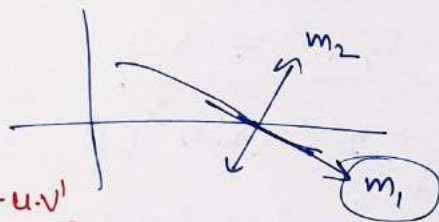
x-axis: $y=0 \Rightarrow 0 = \frac{(x-7)}{(x-2)(x-3)} \Rightarrow 0 = (x-7)$
 $(x=7)$

Point P(7,0)

$y = \frac{(x-7)}{x^2 - 5x + 6}$

$\left(\frac{u}{v}\right)'$

$\frac{u'v - u \cdot v'}{v^2}$



$\frac{dy}{dx} =$

$\frac{1(x^2 - 5x + 6) - (x-7)(2x-5)}{(x^2 - 5x + 6)^2}$

$\left. \frac{dy}{dx} \right|_{(7,0)} = \frac{(\cancel{49} - \cancel{35} + 6) - (\cancel{7} - \cancel{7})(2x-5)}{(49 - 35 + 6)^2} = \frac{1}{20}$

$\left. \frac{dy}{dx} \right|_{(7,0)} = \frac{1}{20} = \text{slope of tangent}$

$-20 = \text{slope of normal}$

$\frac{1}{m_1 m_2} = -1$

Point (7,0)

(7.0) $(y - y_0) = m(x - x_0)$

Equation of Tangent

$$(y - 0) = \left(\frac{1}{20}\right) \cdot (x - 7)$$

Equation of Normal.

$$(y - 0) = (-20) \cdot (x - 7)$$

e.g. Find the equation of tangent to the curve given by $x = a \sin^3 t$, $y = b \cos^3 t$ at a point $t = \frac{\pi}{2}$.
(Parametric Form)

Eqⁿ of Tangent $(y - y_0) = \frac{dy}{dx} \Big|_{(x_0, y_0)} \cdot (x - x_0)$

Point $(x_0, y_0) \equiv (a, 0)$

$t = \frac{\pi}{2} \rightarrow x = a \sin^3\left(\frac{\pi}{2}\right) = a$ Point $(a, 0)$
 $y = b \cos^3\left(\frac{\pi}{2}\right) = 0$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{d(b \cos^3 t)}{d(a \sin^3 t)} = \frac{b \cdot \cancel{3} (\cos t)^2 \cdot (-\sin t)}{a \cdot \cancel{3} (\sin t)^2 \cdot (\cos t)}$$

$$\frac{dy}{dx} = -\frac{b (\cot t)}{a}$$

$\frac{dy}{dx} \Big|_{(a, 0)} = -\frac{b}{a} \cot\left(\frac{\pi}{2}\right) = 0 = \text{slope of tangent}$

Equation of Tangent $(y - y_0) = m(x - x_0)$

$(a, 0)$ $(y - 0) = 0(x - a)$

$y - 0 = 0 \Rightarrow y = 0$

Exercise 6.3 Chapter 6

Tangents and Normals.

At Point 'P' (x_0, y_0)

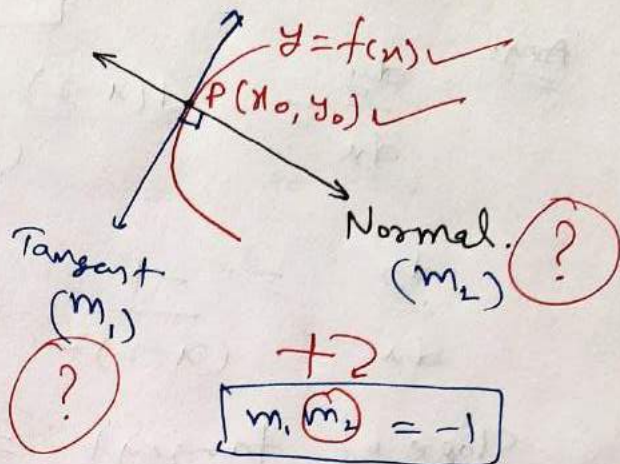
Tangent \parallel Curve \perp Normal

$$m_1 = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \quad m_2$$

Tan

$$m_1 = \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$$

$$m_2 = \frac{-1}{\left(\left. \frac{dy}{dx} \right|_{(x_0, y_0)} \right)}$$



Exercise 6.3

Q.1 Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x=4$.

Ans:

$$\text{Slope of curve} = \frac{dy}{dx} = 12x^3 - 4$$

$$\text{Slope of curve at } (x=4) = \left. \frac{dy}{dx} \right|_{x=4} = 12(4)^3 - 4$$

$$= 12 \times 64 - 4 = 764$$

Slope of tangent at $(x=4) = 764$

Q.2 Find the slope of the tangent to the curve ~~to the~~ $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x=10$

Ans.

$$\frac{dy}{dx} = \frac{1(x-2) - (x-1) \cdot (1)}{(x-2)^2} = \frac{x-2 - x+1}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} \quad (\text{Curve})$$

$$\text{Slope of tangent at } x=10 = \text{Slope of curve at } x=10 = \left. \frac{-1}{(x-2)^2} \right|_{x=10}$$

$$= \frac{-1}{8^2} = \frac{-1}{64} \checkmark$$

Q.3 Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point whose x -coordinate is 2.

$x=2$

Ans.

$$\frac{dy}{dx} = 3x^2 - 1$$

$$\text{Slope of tangent (at } x=2) = \text{Slope of curve (at } x=2) = \left. 3x^2 - 1 \right|_{x=2}$$

$$= 3(2^2) - 1$$

$$= 11 \checkmark$$

Q.4 Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x -coordinate is 3.

Same as Q1, Q2, Q3

Q.5 Find the slope of the normal to the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$.
(Parametric form.)

Ans. $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\left(\frac{d(a \sin^3 \theta)}{d\theta}\right)}{\left(\frac{d(a \cos^3 \theta)}{d\theta}\right)} = \frac{a \cdot 3 \sin^2 \theta \cdot \cancel{\cos \theta}}{a \cdot 3 \cos^2 \theta \cdot (-\cancel{\sin \theta})}$

$$\frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{4}} = -\tan\left(\frac{\pi}{4}\right) = -1 = m_1$$

Normal + Tangent
(m_2) (m_1)

$$m_2 \cdot m_1 = -1$$

$$m_2 = -\frac{1}{m_1}$$

Slope of normal =

$$= \frac{1}{\left(\text{Slope of tangent}\right)} = \frac{1}{(-1)} = 1$$

$$\left(\text{Slope of tangent}\right) = \left(\text{Slope of curve}\right)$$

[Q.6] Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.

Ans.

(Parametric form)

(Point)

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{d(b \cos^2 \theta)}{d(1 - a \sin \theta)} = \frac{b \cdot 2 \cos \theta \cdot (-\sin \theta)}{0 - a \cos \theta}$$

$$\frac{dy}{dx} = \frac{2b \sin \theta}{a} = \text{slope of curve} = \text{slope of tangent.}$$

At $\theta = \frac{\pi}{2}$

$$\text{slope of tangent } (m_1) = \frac{2b \sin \frac{\pi}{2}}{a} = \frac{2b}{a}$$

$$\text{slope of Normal} = -\frac{1}{m_1} = -\frac{1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

(x, y)

[Q.7] Find the points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Ans. If tangent is parallel to the x-axis then slope of tangent = 0

$$y = x^3 - 3x^2 - 9x + 7$$

$$\text{slope of tangent} = \frac{dy}{dx} = 3x^2 - 6x - 9 = 0 \quad \begin{matrix} \text{(put)} \\ \text{(given)} \end{matrix}$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 0$$

$$\Rightarrow 3(x^2 - 2x - 3) = 0$$

$$\Rightarrow x^2 - 3x + x - 3 = 0$$

$$\Rightarrow x(x-3) + 1(x-3) = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$x = 3, x = -1$$

$$y = x^3 - 3x^2 - 9x + 7$$

$$x = 3$$

$$y = \frac{3^3}{7} - 3(3)^2 - 9 \times 3 + 7$$

$$y = -20$$

Point (3, -20)

$$x = -1 \Rightarrow y = (-1)^3 - 3(-1)^2 - 9(-1) + 7$$

$$\Rightarrow y = -1 - 3 + 9 + 7 = 12$$

Point (-1, 12)

Q.8 Find a point on the curve $y = (x-2)^2$ at which tangent is parallel to the chord joining the points (2, 0) and (4, 4)

Ans. Slope of Chord = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{4 - 2} = \frac{4}{2} = 2$

chord \parallel tangent \Rightarrow (Slope of tangent = 2)

$$y = (x-2)^2$$

Slope of tangent = $\frac{dy}{dx} = 2(x-2)$

$$2(x-2) = 2$$

$$\Rightarrow x-2 = 1$$

$$\Rightarrow x = 3$$

$$y = (x-2)^2$$

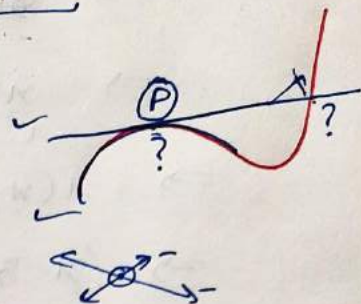
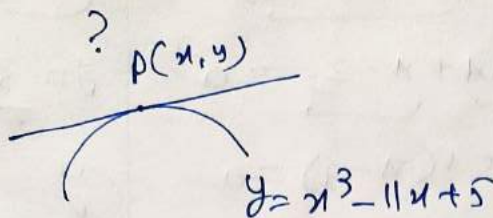
$$y = (3-2)^2 = 1$$

Point (3, 1)

Q.9 Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Ans.

$y = x - 11$
(Tangent)



Line $y = mx + c$
↑
(Slope of this line)

$y = x - 11$ tangent
↓
 $y = 1 \cdot x - 11$
↑
Slope = 1
of tangent

$y = x^3 - 11x + 5$
Slope of tangent = $\frac{dy}{dx}$
 $= 3x^2 - 11$

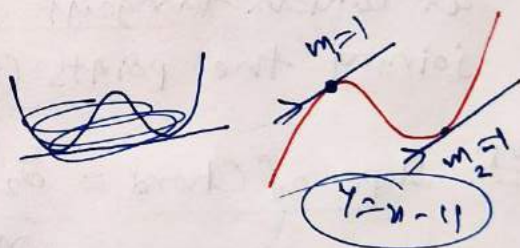
By eqⁿ ① & ② →

$$1 = 3x^2 - 11$$

$$\Rightarrow 12 = 3x^2$$

$$\Rightarrow 4 = x^2$$

$$\Rightarrow x = \pm 2$$



$$x = 2$$

$$x = -2$$

$y = x^3 - 11x + 5$
Curve

$$y = 2^3 - 11(2) + 5$$

$$y = 8 - 22 + 5$$

$$y = -9$$

Point $(2, -9)$

$$y = -8 + 22 + 5$$

$$y = 19$$

Point $(-2, 19)$

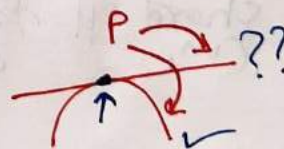
Point ~~X~~

Tangent $y = x - 11$

$(2, -9)$ $-9 = 2 - 11$ ✓

$(-2, 19)$ $19 \neq -2 - 11$

~~X~~ $19 \neq -13$



Exercise 6.3 Chapter-6

(Tangents & Normals)

Q.10 Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}$, $x \neq 1$.

Ans. Line \rightarrow Point (x_0, y_0) ?
 \downarrow Slope (m)

$(y - y_0) = m(x - x_0)$

$y = \frac{1}{x-1}$

by diff.

$\frac{dy}{dx} = \frac{-1}{(x-1)^2}$ = slope of tangent = m_T

Given Slope of tangent = -1

$-1 = \frac{-1}{(x-1)^2}$

$\Rightarrow (x-1)^2 = 1$

$\Rightarrow (x-1) = \pm 1 \Rightarrow x = 1 \pm 1$

$x = 2, x = 0$ x-coordinates

$y = \frac{1}{x-1}$ $(x=2) \rightarrow y = \frac{1}{2-1} = 1$ (2, 1)

$y = \frac{1}{x-1}$ $(x=0) \rightarrow y = \frac{1}{0-1} = -1$ (0, -1)

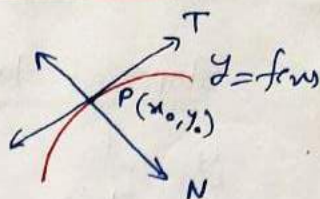
$m = -1$, Point (2, 1)

$(y-1) = -1(x-2) \Rightarrow y = -x + 3$

$m = -1$, Point (0, -1)

$(y+1) = -1(x-0) \Rightarrow y+1 = -x$

$y+x+1=0$



Curve \parallel Tangent \perp Normal

$m_C = m_T$

m_N

$\frac{dy}{dx} \Big|_{(x_0, y_0)}$

$m_N = \frac{-1}{m_T}$

Q.11 Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$, $x \neq 3$.

Ans. Slope of tangent = 2 (given) ✓ — (1)

$$y = \frac{1}{x-3}$$

Slope of tangent = $\frac{dy}{dx} = \frac{-1}{(x-3)^2}$ ✓ — (2)

By eqⁿ (1) & (2) $-\frac{1}{(x-3)^2} = 2$

$$\Rightarrow \left(-\frac{1}{2}\right) = (x-3)^2$$

$$\Rightarrow \pm \sqrt{-\frac{1}{2}} = (x-3)$$

imaginary No.

Not possible

Q.12 Find the equations of all lines having slope '0' which are tangent to the curve $y = \frac{1}{(x^2-2x+3)}$.

Ans. Slope of tangent = 0 (given) — (1)

Slope of tangent = $\frac{dy}{dx} = \frac{-1}{(x^2-2x+3)^2} \cdot (2x-2)$ — (2)

By eqⁿ (1) & (2):

$$\Rightarrow \frac{-2(x-1)}{(x^2-2x+3)^2} = 0$$

$$\Rightarrow x-1=0$$

$$\boxed{x=1}$$

$$y = \frac{1}{x^2-2x+3}$$

Put $x=1$

$$y = \frac{1}{1-2+3} = \frac{1}{2}$$

$$\left(1, \frac{1}{2}\right)$$

$$\left(1, \frac{1}{2}\right), m=0$$

$$(y-y_0) = m(x-x_0)$$

$$\boxed{y - \frac{1}{2} = 0}$$

Tangent

Q.13 Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are (i) Parallel to x-axis.
(ii) Parallel to y-axis.

Ans. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

by diff. w.r.t. 'x'

$$\Rightarrow \frac{2x}{9} + \frac{2y}{16} \cdot \left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{16x}{9y}} = m_T$$

(i) tangents || x-axis.

$$m_T = 0$$

$$\Rightarrow \frac{-16x}{9y} = 0$$

$$\Rightarrow \boxed{x=0}$$

x-coordinate

$$\left. \begin{array}{l} \frac{x^2}{9} + \frac{y^2}{16} = 1 \\ \text{put } x=0 \end{array} \right\}$$

$$\frac{y^2}{16} = 1$$

$$\boxed{y = \pm 4}$$

Points $(0, \pm 4)$

(ii) tangents || y-axis

Slope of tangents = ∞
 $m_T = \infty = \frac{1}{0}$

$$\Rightarrow \frac{-16x}{9y} = \frac{1}{0}$$

$$\Rightarrow 0 = 9y$$

$$\Rightarrow \boxed{y=0}$$

y-coordinate

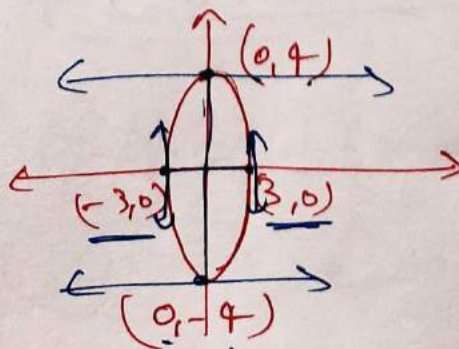
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

put y=0

$$\Rightarrow \frac{x^2}{9} = 1$$

$$\Rightarrow \boxed{x = \pm 3}$$

Points $(\pm 3, 0)$



$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Q.14 Find the equations of Tangents and Normals.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$
at $(0, 5)$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$
at $(1, 3)$

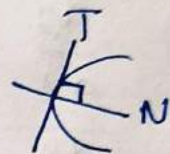
Slope of tangent $= \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$
at $(0, 5)$

Slope of tangent at $(0, 5) = m_T = 0 - 0 + 0 - 10$

$m_T = -10$

$m_N \cdot m_T = -1$

$m_N = -\frac{1}{m_T} = -\frac{1}{-10} = \frac{1}{10}$



$(y - y_0) = m(x - x_0)$

Equation of tangent at $(0, 5)$

$(y - 5) = (-10)(x - 0) \Rightarrow y - 5 = -10x$
 $\Rightarrow \boxed{10x + y = 5}$

Equation of Normal at $(0, 5)$

$(y - 5) = \frac{1}{10}(x - 0)$

$\Rightarrow 10y - 50 = x$

$\Rightarrow \boxed{0 = x - 10y + 50}$

(iii) $y = x^3$ at $(1,1)$

$$\frac{dy}{dx} = 3x^2$$

Slope of tangent at $(1,1) = m_T = 3(1)^2 = 3$ ✓

Slope of normal at $(1,1) = m_N = -\frac{1}{3}$ ✓

Equation $(1,1)$

Tangent, $(y-1) = 3(x-1) \Rightarrow y-1 = 3x-3 \Rightarrow y = 3x-2$

Normal $(y-1) = -\frac{1}{3}(x-1) \Rightarrow 3y-3 = -x+1 \Rightarrow x+3y-4=0$

(iv) $y = x^2$ at $(0,0)$

$$\frac{dy}{dx} = 2x$$

Tangent's slope

$$m_T = 0$$

$$y = k$$

Normal's slope

$$m_N = -\frac{1}{0} = \infty, -\infty$$

$$x = k$$

Equation, $(y-y_0) = m(x-x_0)$

Tangent, $m_T = 0$

$$y-0 = 0(x-0)$$

$$y-0 = 0 \Rightarrow y=0$$

Normal, $m_N = \frac{1}{0}$

$$(y-0) = \frac{1}{0}(x-0)$$

$$0 = x$$

(5) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$

Parametric Form

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{d(\sin t)}{d(\cos t)} = \frac{\cos t}{-\sin t} = -\cot t$$

Slope of tangent at $t = \frac{\pi}{4} = m_T = -\cot(t)$
 $= -\cot\left(\frac{\pi}{4}\right)$
 $= -1$

$m_N = -\frac{1}{-1} = +1$

Equation: $(y - y_0) = m(x - x_0)$

Tangent

$$\left(y - \frac{1}{\sqrt{2}}\right) = -1\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow y - \frac{1}{\sqrt{2}} = -x + \frac{1}{\sqrt{2}}$$

$$\Rightarrow x + y - \frac{2}{\sqrt{2}} = 0$$

$$\Rightarrow \boxed{x + y - \sqrt{2} = 0}$$

Normal: $y - \frac{1}{\sqrt{2}} = +1\left(x - \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow y - \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$$

$$\boxed{y = x}$$

Point $(x_0, y_0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

x_0	y_0
\downarrow	\downarrow
$\cos \frac{\pi}{4}$	$\sin \frac{\pi}{4}$
\downarrow	\downarrow
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{2}}$$

Exercise 6.3 Chapter 6

Q.15 Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

Ans. $y = x^2 - 2x + 7$

$$\frac{dy}{dx} = 2x - 2 = m_T = \text{slope of tangent.}$$

$$m_T = 2(x - 1)$$

$$y = mx + c$$

slope

(a) Tangent $\parallel 2x - y + 9 = 0$

\downarrow \downarrow

$$m_T = m_1 = 2$$

$$y = 2x + 9$$

$$\Rightarrow 2(x - 1) = 2$$

$$\Rightarrow x - 1 = 1$$

$$\Rightarrow \boxed{x = 2}$$

x-coordinate.

$$x = 2 \rightarrow \text{put} \rightarrow y = x^2 - 2x + 7$$

$$y = 2^2 - 2 \cdot 2 + 7$$

$$\text{Point } (2, 7)$$

Tangent $(y - y_0) = m_T(x - x_0)$

$$(y - 7) = 2(x - 2) \Rightarrow y - 7 = 2x - 4$$

$$\Rightarrow \boxed{y = 2x + 3}$$

(b) (Tangent) $\perp (5y - 15x = 13) \rightarrow 5y = 15x + 13$

\downarrow

$$m_T$$

\downarrow

$$m_2$$

$$y = 3x + 13/5$$

$$m_T \cdot m_2 = -1$$

$$\boxed{m_T = -\frac{1}{m_2}} \Rightarrow \boxed{m_T = -\frac{1}{3}} = 2(x - 1)$$

$$-\frac{1}{3} = 2(x-1)$$

$$y = x^2 - 2x + 7$$

$$\Rightarrow -\frac{1}{6} = x-1$$

$$y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7$$

$$\Rightarrow 1 - \frac{1}{6} = x$$

$$y = \frac{25}{36} - \frac{5}{3} + \frac{7}{1} = \frac{25 - 60 + 252}{36}$$

$$\Rightarrow \boxed{x = \frac{5}{6}}$$

$$y = \frac{217}{36}$$

$$m_T = -\frac{1}{3}$$

Tangent $(y - y_0) = m_T (x - x_0)$

$$\Rightarrow \left(y - \frac{217}{36}\right) = -\frac{1}{3} \left(x - \frac{5}{6}\right)$$

$$\Rightarrow \frac{36y - 217}{36} = -\frac{1}{3} \left(\frac{6x - 5}{6}\right)$$

$$\Rightarrow 36y - 217 = -12x + 10$$

$$\Rightarrow \boxed{12x + 36y = 227}$$

Q.16 Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x=2$ and $x=-2$ are parallel.

Ans. $y = 7x^3 + 11$

Slope of tangent $= \frac{dy}{dx} = 21x^2$

Slope of tangent at $(x=2) = 21(2)^2 = 84 = m_1$

Slope of tangent at $(x=-2) = 21(-2)^2 = 84 = m_2$

$$m_1 = m_2$$

$x=2$ Tangent

$x=-2$ Tangent

Q.17 Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Ans. Slope of tangent = $m_T = \frac{dy}{dx} = 3x^2$

$$m_T = 3x^2$$

Let Point be (x, y)

↓
Satisfy

ATQ. $3x^2 = y$ — (1)

$$y = x^3$$
 — (2)

By eqⁿ (1) & (2)

$$3x^2 = x^3$$

$$\Rightarrow 0 = x^3 - 3x^2$$

$$\Rightarrow 0 = x^2(x - 3)$$

$$x = 0$$

$$x = 3$$

$$y = x^3$$

$$x = 0 \Rightarrow y = 0^3 \Rightarrow y = 0 \quad (0, 0)$$

$$x = 3 \Rightarrow y = 3^3 \Rightarrow y = 27 \quad (3, 27)$$

Q.18 For the curve $y = 4x^3 - 2x^5$, Find all the points at which the tangent passes through the origin.

Ans. Let the point be (x_0, y_0) P.

$$y_0 = 4x_0^3 - 2x_0^5$$
 — (1)

Tangent at $P(x_0, y_0)$, $(y - y_0) = m_T(x - x_0)$

Slope of tangent = $m_T = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} = (12x^2 - 10x^4) \Big|_{(x_0, y_0)}$

$$m_T = 12x_0^2 - 10x_0^4$$

$$y = 4x^3 - 2x^5$$

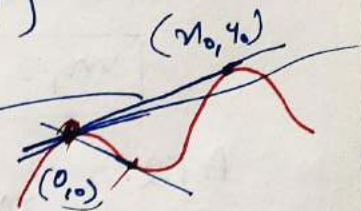
$$\boxed{y_0 = 4x_0^3 - 2x_0^5} \quad \text{--- (1)}$$

Equation of tangent $(y - y_0) = m_T (x - x_0)$

$$\uparrow$$

$$(12x_0^2 - 10x_0^4)$$

$$\Rightarrow \boxed{(y - y_0) = (12x_0^2 - 10x_0^4) \cdot (x - x_0)}$$



This tangent passes through the origin

$$x = 0$$

$$y = 0$$

(0, 0)

$$\Rightarrow (0 - y_0) = (12x_0^2 - 10x_0^4) \cdot (0 - x_0)$$

$$\Rightarrow \boxed{y_0 = 12x_0^3 - 10x_0^5} \quad \text{--- (2)}$$

$$\boxed{y_0 = 4x_0^3 - 2x_0^5} \quad \text{--- (1)}$$

By eqn (1) & (2):

$$\Rightarrow \underline{12x_0^3} - \underline{10x_0^5} = 4x_0^3 - 2x_0^5$$

$$\Rightarrow 8x_0^5 - 8x_0^3 = 0$$

$$\Rightarrow \cancel{8} x_0^3 (x_0^2 - 1) = 0$$

$$\Rightarrow \boxed{x_0 = 0}, \quad x_0^2 = 1$$

$$\boxed{x_0 = \pm 1}$$

Point
(x_0, y_0)
?

$$y_0 = 4x_0^3 - 2x_0^5$$

$$\boxed{x_0 = 0}, \quad \boxed{y_0 = 0} \quad (0, 0) \quad \checkmark$$

$$\boxed{x_0 = 1}, \quad \boxed{y_0 = 2} \quad (1, 2) \quad \checkmark$$

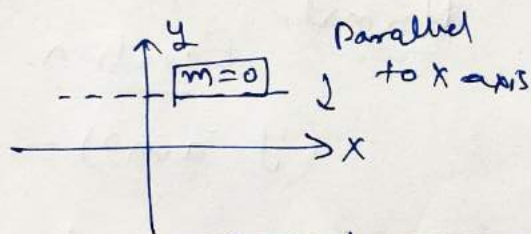
$$\boxed{x_0 = -1}, \quad \boxed{y_0 = -2} \quad (-1, -2) \quad \checkmark$$

(0, 0), (1, 2), (-1, -2)

origin

Q.19 Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Ans.
 $x^2 + y^2 - 2x - 3 = 0$
 by diff. w.r.t. 'x'



tangent || x-axis

$$m_T = 0$$

$$\frac{-x+1}{y} = 0$$

$$\Rightarrow x = 1 \text{ x-coordinate}$$

$$\Rightarrow 2x + 2y \left(\frac{dy}{dx}\right) - 2 = 0$$

$$\Rightarrow 2y \left(\frac{dy}{dx}\right) = -2x + 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x+1}{y} = \text{Slope of tangent}$$

Points

$$(1, \pm 2)$$

$$x^2 + y^2 - 2x - 3 = 0$$

$$\Rightarrow 1 + y^2 - 2 - 3 = 0$$

$$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2 \text{ y-coordinate}$$

Q.20 Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Ans. Normal, $(y - am^3) = m_N \cdot (x - am^2)$

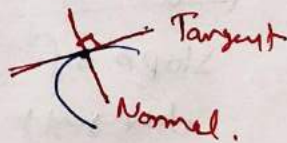
$$m_N \cdot m_T = -1$$

$$m_N = -\frac{1}{m_T}$$

$$ay^2 = x^3$$

by diff. w.r.t. 'x'

$$2ay \left(\frac{dy}{dx}\right) = 3x^2$$



Slope of tangent

$$m_T = \frac{dy}{dx} = \frac{3x^2}{2ay} \Big|_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2 m^4}{2a^2 m^3} = \frac{3m}{2}$$

$$m_N = \frac{-1}{m_T} = \frac{-1}{\left(\frac{3m}{2}\right)} = \frac{-2}{3m}$$

Normal,
Equation.

$$(y - am^3) = \frac{-2}{3m} (x - am^2)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow \underline{2x + 3my} = \underline{3am^4 + 2am^2} = \underline{am^2(3m^2 + 2)}$$

$$\boxed{2x + 3my = am^2(3m^2 + 2)}$$

[Q.21] Find the equation of the normal to the ~~curve~~ curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

Ans. ~~Normal~~ Normal $\parallel x + 14y + 4 = 0$

$$m_N = m$$

$$\boxed{m_N = -\frac{1}{14}}$$

$$y = m x + c$$

$$14y = -x - 4$$

$$y = -\frac{x}{14} + \frac{4}{14}$$

$$\boxed{y = x^3 + 2x + 6}$$

$$\text{Slope of tangent} = \frac{dy}{dx} = 3x^2 + 2 = m_T$$

$$\text{Slope of Normal} = m_N = \frac{-1}{m_T} = \frac{-1}{3x^2 + 2}$$

By eqn ① & ②

$$\frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow \boxed{x = \pm 2}$$

$$y = x^3 + 2x + 6$$

$$\boxed{x=2}$$

$$y = 8 + 4 + 6$$

$$y = 18$$

$$(2, 18)$$

$$m_N = -\frac{1}{14}$$

Eqⁿ. of Normal,

$$(y - 18) = -\frac{1}{14}(x - 2)$$

$$\Rightarrow 14y - 252 = -x + 2$$

$$\Rightarrow \boxed{x + 14y - 254 = 0}$$

$$\boxed{x = -2}$$

$$y = -8 - 4 + 6$$

$$y = -6$$

$$(-2, -6)$$

$$m_N = -\frac{1}{14}$$

Eqⁿ. of Normal,

$$(y + 6) = -\frac{1}{14}(x + 2)$$

$$\Rightarrow 14y + 84 = -x - 2$$

$$\Rightarrow \boxed{x + 14y + 86 = 0}$$

Q.22 Find the equation of tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Ans. $y^2 = 4ax$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 4a$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

Eqⁿ. of tangent,

$$\Rightarrow y - y_0 = m_T(x - x_0)$$

$$\Rightarrow (y - 2at) = \frac{1}{t}(x - at^2)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$

$$\Rightarrow \boxed{ty = x + at^2}$$

Slope of tangent at $(at^2, 2at)$ $= m_T = \frac{2a}{2at} = \frac{1}{t}$ ✓

$$m_N = -\frac{1}{m_T} = -\frac{1}{\left(\frac{1}{t}\right)} = -t \quad \checkmark$$

Eqⁿ. of Normal,

$$(y - 2at) = (-t)(x - at^2)$$

$$\Rightarrow \boxed{y = 2at - xt + at^3}$$

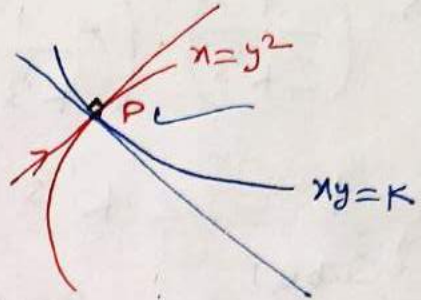
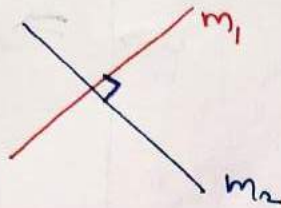
Exercise 6.3

Chapter 6

Q.23

Prove that the curves $x=y^2$ and $xy=k$ cut at right angles if $8k^2=1$.

$m_1 m_2 = -1$



For Point 'P'

$x=y^2$ — (1)

$xy=k$ — (2)

Solve.

$$\begin{aligned} \Rightarrow (y^2)y &= k & \left| \quad x &= (k^{1/3})^2 \\ \Rightarrow y^3 &= k & \left| \quad x &= k^{2/3} \\ \Rightarrow y &= k^{1/3} & \left| \quad \text{Point } P(k^{2/3}, k^{1/3}) \right. & \leftarrow \text{Point of intersection} \end{aligned}$$

$x=y^2$

by diff. w.r.t. 'x'

$\Rightarrow 1 = 2y \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

$m_1 = \text{slope of tangent of } x=y^2 \text{ at 'P'} = \frac{1}{2k^{1/3}}$

$xy=k$

$\Rightarrow y = \frac{k}{x}$

by diff. w.r.t. 'x'

$\Rightarrow \frac{dy}{dx} = -\frac{k}{x^2}$

$m_2 = \text{slope of tangent of } xy=k \text{ at 'P'} = \frac{-k}{(k^{2/3})^2} = \frac{-k}{k^{4/3}} = -\frac{1}{k^{1/3}}$

we have $m_1 = \frac{1}{2k^{1/3}}$, $m_2 = -\frac{1}{k^{1/3}}$

Condition of perpendicularity.

$$m_1 m_2 = -1$$

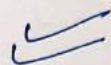
$$\Rightarrow \left(\frac{1}{2k^{1/3}}\right) \cdot \left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow 1 = 2 \cdot k^{2/3}$$

(Cube)

$$\Rightarrow 1^3 = (2k^{2/3})^3$$

$$\Rightarrow 1 = 8k^2$$



Q. 24 Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) .

Ans. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

by diff. w.r.t 'x'

$$\Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2} \left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{b^2} \left(\frac{dy}{dx}\right)$$

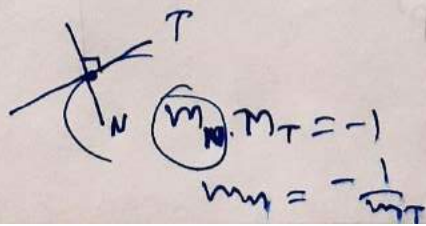
$$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Slope of tangent at (x_0, y_0)

$$m_T = \frac{b^2 x_0}{a^2 y_0}$$

Slope of Normal at (x_0, y_0)

$$m_N = -\frac{1}{m_T} = -\frac{a^2 y_0}{b^2 x_0}$$



Equation of Tangent

$$(y - y_0) = \frac{b^2 x_0}{a^2 y_0} (x - x_0)$$

$$\Rightarrow \frac{y y_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x x_0}{a^2} - \frac{x_0^2}{a^2}$$

$$\Rightarrow \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = \frac{x x_0}{a^2} - \frac{y y_0}{b^2}$$

$$\Rightarrow \boxed{1 = \frac{x x_0}{a^2} - \frac{y y_0}{b^2}} \quad \text{Eqn. of tangent}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(x_0, y_0) lies on

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$$

Equation of Normal

$$(y - y_0) = -\frac{a^2 y_0}{b^2 x_0} (x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} = -\frac{(x - x_0)}{b^2 x_0}$$

$$\Rightarrow \boxed{\frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0}$$

Q.25 Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Ans. Tangent \parallel $4x - 2y + 5 = 0$

\downarrow \downarrow
 $m_T = m$
 \downarrow
 $m_T = 2$ ①

$y = mx + c$

$2y = 4x + 5$
 $y = 2x + \frac{5}{2}$
 \downarrow
 $m = 2$

$y = \sqrt{3x-2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}} \cdot (3) = \text{slope of tangent} = m_T$ ②

By eqⁿ ① & ② \rightarrow

$2 = \frac{3}{2\sqrt{3x-2}}$

$\Rightarrow 4\sqrt{3x-2} = 3$

$\Rightarrow 16(3x-2) = 9$

$\Rightarrow \underline{3x-2 = \frac{9}{16}}$

$\Rightarrow 3x = \frac{41}{16}$

$\Rightarrow \boxed{x = \frac{41}{48}}$

$y = \sqrt{3x-2}$

$x = \frac{41}{48}$ ✓

$y = \sqrt{\frac{9}{16}} = \frac{3}{4}$ ✓

Point $\left(\frac{41}{48}, \frac{3}{4}\right)$, $m_T = 2$

Equation of tangent.

$(y - \frac{3}{4}) = 2(x - \frac{41}{48})$

$\Rightarrow \frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$

$\Rightarrow 24y - 18 = 48x - 41$

$\boxed{23 = 48x - 24y}$

Q.26 The slope of the normal to the curve

$y = 2x^2 + 3\sin x$ at $x=0$ is -

- (A) 3 (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{1}{3}$

Ans. $y = 2x^2 + 3\sin x$

$\frac{dy}{dx} = 4x + 3\cos x =$ Slope of tangent.

Slope of tangent at $(x=0) = 4(0) + 3\cos(0)$
 $= 0 + 3 = 3$

Slope of normal at $(x=0) = \frac{-1}{3}$

~~$m_1 m_2 = -1$~~

tangent's slope

Q.27 The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

- (A) (1,2) (B) (2,1) (C) (1,-2) (D) (-1,2)

Ans.

$y^2 = 4x$ (2) Q.9 $y^2 = 4x$

$y = x + 1$ (1)

By solving both equations

$y = x + 1$ & $y^2 = 4x$ for point P

$\Rightarrow (x+1)^2 = 4x$

$\Rightarrow x^2 + 2x + 1 = 4x$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x - 1 = 0$

$x = 1$

$y = x + 1$

$y = 1 + 1$

$y = 2$

Point

(1,2)

Approximations:

(सन्निकटन)

$$\begin{array}{cc} \sqrt{100} & \sqrt{101} \\ \downarrow & \downarrow \\ 10 & \approx 10.1 \\ & \approx 10.01 \\ & \approx \underline{10.049} \text{ (Calc)} \end{array}$$

$$y = f(x) = \sqrt{x}$$

$$\sin 30^\circ = \frac{1}{2} = 0.5$$

$$\sin(30.5^\circ) \approx \frac{1}{2}$$

$$\hookrightarrow 0.501$$

$$y = f(x) = \sin x$$

Derivation.

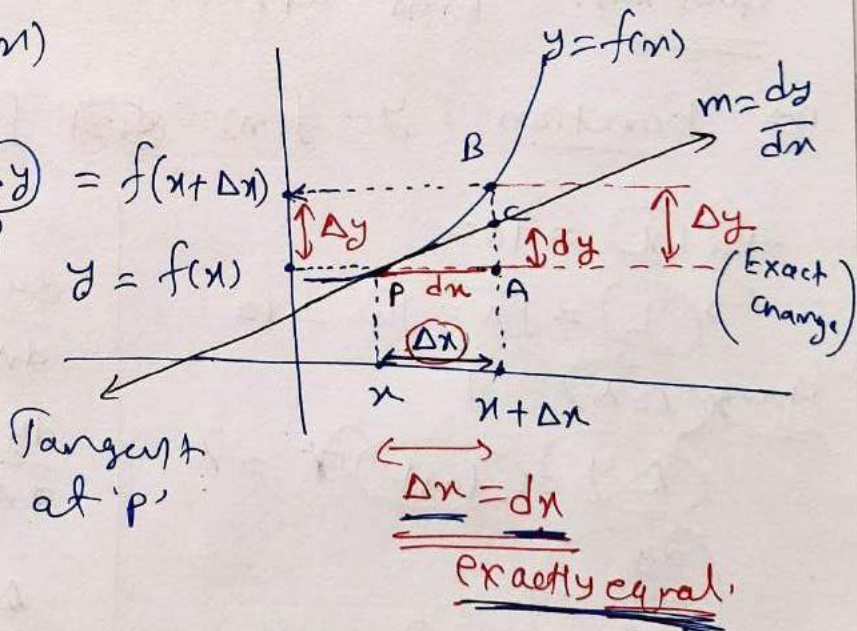
$$y = f(x)$$

$$y + \Delta y = f(x + \Delta x)$$

$$y = f(x)$$

input x
 \downarrow
 no
Approximation

Output y and values
 \downarrow
approximation



Approximation,

$$AC \approx AB \Rightarrow$$

$$AC = AB \text{ approx}$$

$$dy \approx \Delta y$$

$$y = f(x)$$

$$\frac{y + \Delta y}{?} = f(x + \Delta x)$$

$$dx = \Delta x$$

$$dy \approx \Delta y$$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x}$$

(approximately)

$$\Rightarrow \left[\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right] \Rightarrow \Delta y = \frac{dy}{dx} \cdot \Delta x$$

Question
(Given)

by diff.

(Approximately)

Question: Find approximate value of $\sqrt{101}$.

Ans: Function $y = f(x) = \sqrt{x}$

$$\begin{array}{c} \sqrt{100+1} \\ \uparrow \quad \uparrow \\ \sqrt{x+\Delta x} \end{array}$$

Let $x = 100$

$y = \sqrt{x} = \sqrt{100} = 10$

change $\Delta x = 1$

$\Delta y = \text{last digit} = ?$

$\frac{dy}{dx} = \frac{1}{20}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{100}} = \frac{1}{2 \times 10} = \frac{1}{20}$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ (approx.)}$$

$$\frac{\Delta y}{1} = \frac{1}{20}$$

$$\begin{array}{l} \sqrt{x} = y \\ \sqrt{100} = 10 \end{array}$$

$$\begin{array}{l} \sqrt{x+\Delta x} = y+\Delta y \\ \sqrt{100+1} = 10 + \frac{1}{20} \\ = 10.05 \end{array}$$

50.01

e.g. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Ans.

$$V = \frac{4}{3} \pi r^3$$



$r = 9 \text{ cm}$
 $\Delta r = 0.03 \text{ cm}$
 $\rightarrow V = \frac{4}{3} \pi (9^3)$
 $\Delta V = ? = \text{approx. error in volume}$
 $\frac{dV}{dr} = 4\pi(9^2)$

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3} \pi (3r^2)$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$= 4\pi(9)^2$$

Approximation,

$$\frac{\Delta V}{\Delta r} = \frac{dV}{dr}$$

$$\Rightarrow \Delta V = \frac{dV}{dr} \cdot \Delta r$$

$$\Rightarrow \Delta V = 4\pi(9)^2 \cdot (0.03)$$

Approximate error in Volume

$$\Delta V = 9.72\pi \text{ cm}^3$$

Approximate volume = $V + \Delta V$

$$= \frac{4}{3} \pi (9)^3 + (9.72\pi)$$

Exercise 6.4

Chapter 6

Approximation

$$y = f(x)$$

$$y + \Delta y = f(x + \Delta x)$$

?

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

Exercise 6.4

Q.1 → (ii), (v), (xiv)

Q.2 Q.3 Q.8

Q.4 Q.5 Q.9

Q.6 Q.7

← Questions with same type.

← Same type.

← Same type.

Exercise-6.4 (Approximation)

Q.1

(ii) $\sqrt{49.5}$ function $y = \sqrt{x}$

$x = 49$, $y = \sqrt{x} = \sqrt{49} = 7$

$\sqrt{49.5}$	$y + \Delta y = \sqrt{x + \Delta x}$
$49.5 = x + \Delta x$	$= \sqrt{49.5}$
$\Delta x = 0.5$	

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{49}}$$

$$= \frac{1}{2 \times 7} = \frac{1}{14}$$

Approx. → $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$\Delta y = \frac{1}{14} \times 0.5 = 0.035$$

$$\sqrt{49.5} = y + \Delta y$$

$$= 7 + 0.035$$

$$= 7.035$$

Hints.

(i) $\sqrt{25.3} \rightarrow y = \sqrt{x}$

(iii) $\sqrt{0.6} \rightarrow y = \sqrt{x}$

(iv) $(0.009)^{1/3} \rightarrow y = x^{1/3}$

$25.3 = x + \Delta x = 25 + 0.3$

$0.6 = 0.60 = \frac{0.64}{x} - \frac{0.04}{\Delta x}$

$0.009 = \frac{0.008}{x} + \frac{0.001}{\Delta x}$

(v) $(0.999)^{1/10}$

$y = x^{1/10}$

$0.999 = \frac{1}{x} - \frac{0.001}{\Delta x}$

$x = 1, y = (1)^{1/10} = 1$

$x + \Delta x = 0.999$
 \downarrow
 $- 0.001$

$\Delta x = -0.001$

$y + \Delta y = (0.999)^{1/10}$
 $= (x + \Delta x)^{1/10}$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$\Delta y = \frac{dy}{dx} \cdot \Delta x$

$\Delta y = \frac{1}{10} x(-0.001)$

$\Delta y = -0.0001$

$y = x^{1/10}$
 $\frac{dy}{dx} = \frac{1}{10} \cdot x^{1/10 - 1}$
 $= \frac{1}{10} \cdot x^{-9/10}$
 $= \frac{1}{10 \cdot x^{9/10}}$
 $\frac{dy}{dx} = \frac{1}{10}$

$(0.999)^{1/10} = y + \Delta y$

$= 1 - 0.0001$

$= 0.9999$

Hints.

(VI) $(15)^{1/4} \rightarrow y = x^{1/4} \rightarrow 15 = \underbrace{16}_{x} - \underbrace{1}_{\Delta x} \quad \underbrace{x=16, \Delta x=-1}$

(VII) $(26)^{1/3} \rightarrow y = x^{1/3} \rightarrow 26 = \underbrace{27}_{x} - \underbrace{1}_{\Delta x}$

(VIII) $(255)^{1/4} \rightarrow y = x^{1/4} \rightarrow 255 = \underbrace{256}_{x} - \underbrace{1}_{\Delta x}$

(IX) $(82)^{1/4} \rightarrow y = x^{1/4} \rightarrow 82 = \underbrace{81}_{x} + \underbrace{1}_{\Delta x}$

(X) $(401)^{1/2} \rightarrow y = x^{1/2} \rightarrow 401 = \underbrace{400}_{x} + \underbrace{1}_{\Delta x}$

(XI) $(0.0037)^{1/2} \rightarrow y = x^{1/2} \rightarrow 0.0037 = \underbrace{0.0036}_{x} + \underbrace{0.0001}_{\Delta x}$

(XII) $(26.57)^{1/3} \rightarrow y = x^{1/3} \rightarrow 26.57 = \underbrace{27}_{x} - \underbrace{0.43}_{\Delta x}$

(XIII) $(81.5)^{1/4} \rightarrow y = x^{1/4} \rightarrow 81.5 = \underbrace{81}_{x} + \underbrace{0.5}_{\Delta x}$

(XIV) $(3.968)^{3/2} \rightarrow y = x^{3/2}$

~~3~~
 $3.968 = \underbrace{4}_{x} - \underbrace{0.032}_{\Delta x}$

$y + \Delta y = (3.968)^{3/2}$

↓

$y = x^{3/2}$

$y = (4)^{3/2}$

$y = (2)^3$

$y = 8$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$\Delta y = \frac{dy}{dx} \cdot \Delta x$

$= 3 \times (-0.032)$

$= -0.096$

$y = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} \cdot x^{1/2}$

$= \frac{3}{2} \times (4)^{1/2}$

$= \frac{3}{2} \times 2$

$= 3$

$(3.968)^{3/2} = y + \Delta y$

$= 8 - 0.096$

$= 7.904 \checkmark$

Similar Questions 2, 3, (8)

Q.8 If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is —

- (A) 47.66 (B) 57.66 (C) 67.66 ~~(D) 77.66~~

Ans. $y = f(x) = 3x^2 + 15x + 5$ $f(3.02)$

$$3.02 = \underbrace{3}_x + \underbrace{0.02}_{\Delta x}$$

$$y = f(3) = 3(3)^2 + 15 + 5 = 77$$

$$y + \Delta y = f(3.02)$$

$$y = 3x^2 + 15x + 5$$

$$\frac{dy}{dx} = 6x + 15$$

$$= 6(3) + 15$$

$$= 33$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\Rightarrow \Delta y = \left(\frac{dy}{dx}\right) \cdot \Delta x$$

$$\Delta y = 33 \times 0.02$$

$$\Delta y = 0.66$$

$$f(3.02) = y + \Delta y = 77 + 0.66 = 77.66$$

Similar Questions 4, 5, (9)

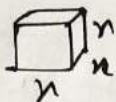
Q.9 The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is —

- (A) $0.06 x^3 \text{ m}^3$ (B) $0.6 x^3 \text{ m}^3$
~~(C) $0.09 x^3 \text{ m}^3$~~ (D) $0.9 x^3 \text{ m}^3$

Ans.

$\Delta V = ?$

Volume of a cube = $V = (\text{side})^3$



(Side = x)

$V = x^3$

Side increased by 3%.

Now side = $x + \Delta x$

$\Delta x = 3\% \text{ of } x$

$= \frac{3}{100} \times x$

$= 0.03x$

Now side = $x + \Delta x = x + 3\% \text{ of } x$

Approxi.

$\frac{\Delta V}{\Delta x} = \left(\frac{dV}{dx} \right) \rightarrow$ By diff.

$\Rightarrow \Delta V = \frac{dV}{dx} \cdot \Delta x$

$\Delta V = (3x^2) \cdot (0.03x)$

$\Delta V = 0.09x^3$

approximate change in volume.

$V = x^3$

$\frac{dV}{dx} = 3x^2$

$V \pm \Delta V$

Similar Questions (6) & 7

(11)

Q.6 If the radius of a sphere is measured as 7m with an error 0.02m, then find the approximate error in calculating its volume.

Ans: Sphere \rightarrow volume = $V = \frac{4}{3} \pi r^3$

(11)

$r = 7\text{ m}$
 $\Delta r = 0.02\text{ m}$

Approximate volume = $V \pm \Delta V$

Approximate error in volume = ΔV

Approximation

$$\frac{\Delta V}{\Delta r} = \frac{dV}{dr}$$

$$\Rightarrow \Delta V = \left(\frac{dV}{dr} \right) \cdot \Delta r$$

$$\Rightarrow \Delta V = 4\pi(49) \cdot (0.02)$$

$$\Rightarrow \Delta V = 3.92 \pi \text{ m}^3$$

$$V = \frac{4}{3} \pi r^3$$

by diff. w.r.t. 'r'

$$\frac{dV}{dr} = \frac{4}{3} \pi (3r^2) = 4\pi r^2$$

$$\frac{dV}{dr} = 4\pi (7)^2$$

$\Delta V = \text{Approximate error in volume} = 3.92 \pi \text{ m}^3$

✓

MAXIMA - MINIMA की कहानी

Graphical Meaning of Differentiable, Non-differentiable \rightarrow

Maxima, minima (Extrema) \rightarrow Turning point (local minima, local maxima) \rightarrow Critical point \rightarrow (Ex.) \rightarrow

First order Derivative test to find local extrema \rightarrow (Ex.)

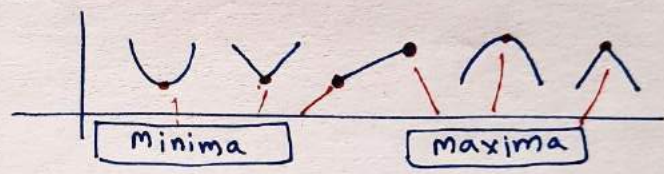
\rightarrow Second order Derivative test to find local extrema \rightarrow

meaning of $f''(x)$ \rightarrow concavity of curve \rightarrow Point of Inflection \rightarrow (Ex.) \rightarrow (Ex.) \rightarrow Absolute (global) minima/maxima

in closed interval \rightarrow (Ex.) \rightarrow (Ex.)

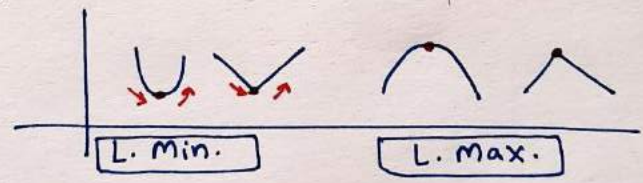
Point of Extrema

(परम बिंदु)



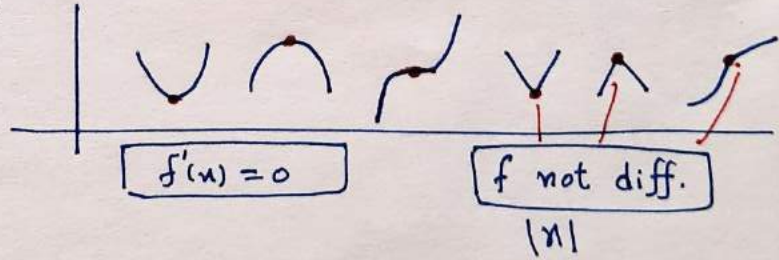
Turning Point

(वर्तन बिंदु)



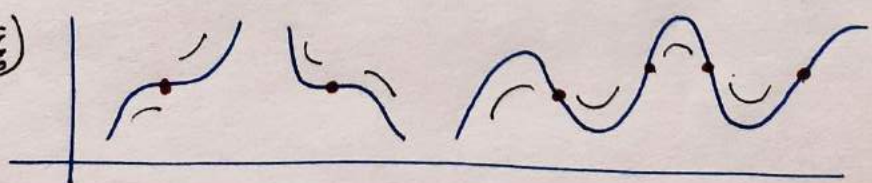
Critical Point

(क्रांतिक बिंदु)



Point of Inflection

(नति परिवर्तन बिंदु)



Maxima and Minima (उच्चतम और निम्नतम) *

Story



$f(x)$

$x = ?$ कहां

Graphical Meaning of Differentiability.

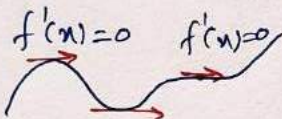
Differentiable
(Smooth)



Non-Differentiable
(Sharp corner)



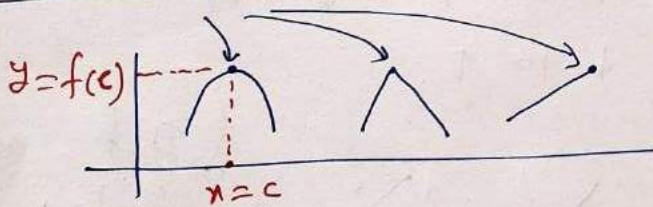
Note:



→ Horizontal
(flat)

$f'(x) = 0 = \text{slope}$

Maxima (maximum) : →

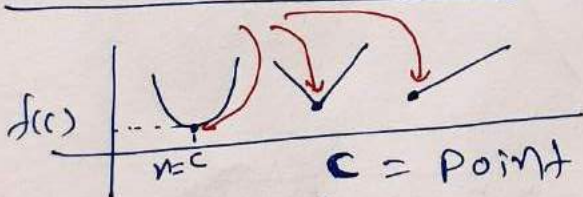


(peaks)

$c =$ Point of maximum value = point of maxima

$f(c) =$ maximum value of 'f'

Minima (minimum) : →

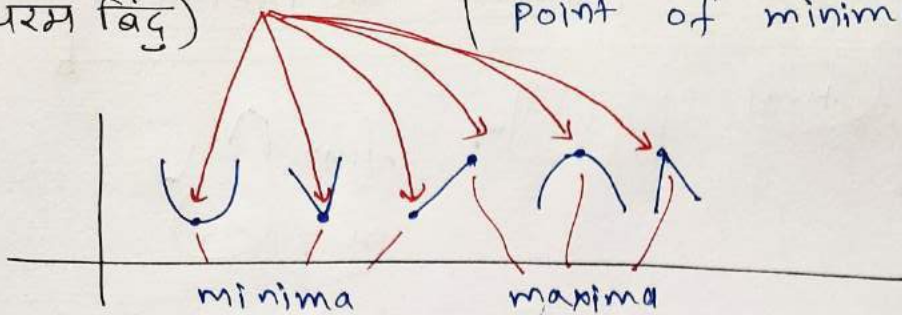


(valley)

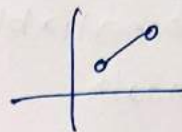
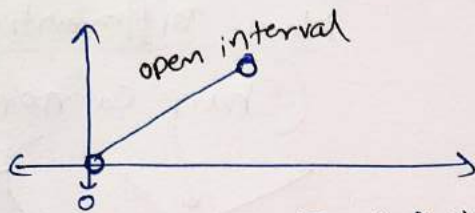
$c =$ point of minima

$f(c) =$ value of ~~value~~ minima of 'f'

Point of Extrema \rightarrow $\left\{ \begin{array}{l} \text{Point of maxima} \\ \text{Point of minima} \end{array} \right\}$
 (उत्तम बिंदु)



Note:



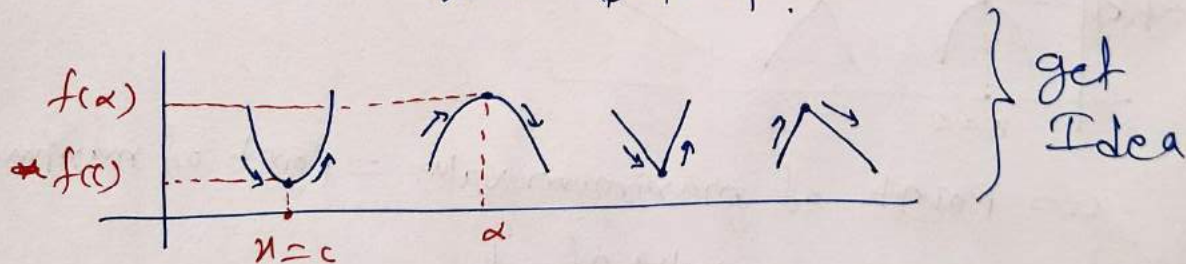
$$\frac{0.001}{2}$$

no minima
no maxima

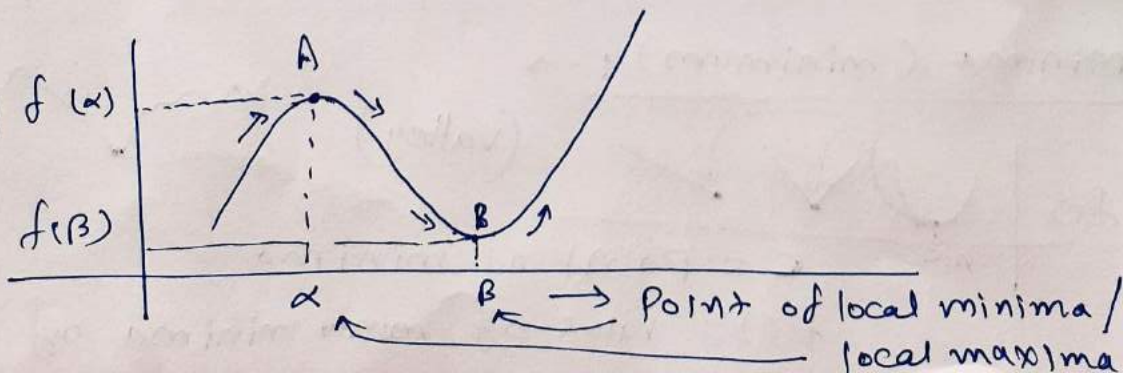
no extrema

$$= \frac{0.0005}{2} \quad 0.00025$$

Turning Point : where function changes its nature from \uparrow to \downarrow or from \downarrow to \uparrow .
 (वर्तन बिंदु)



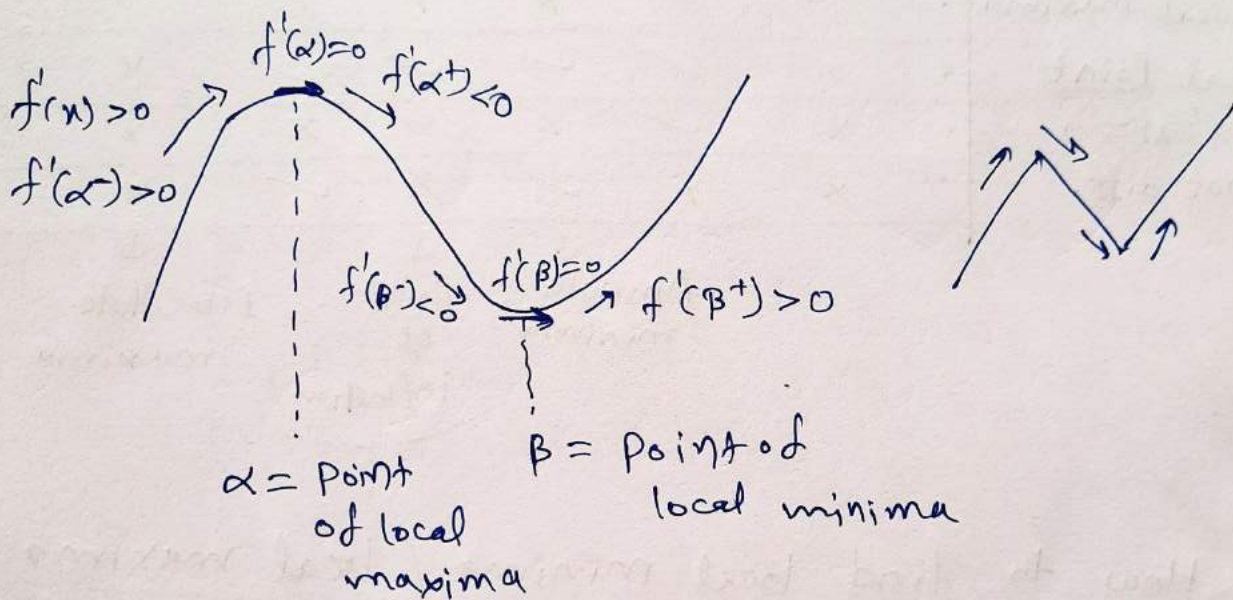
local maximum value
local minimum value



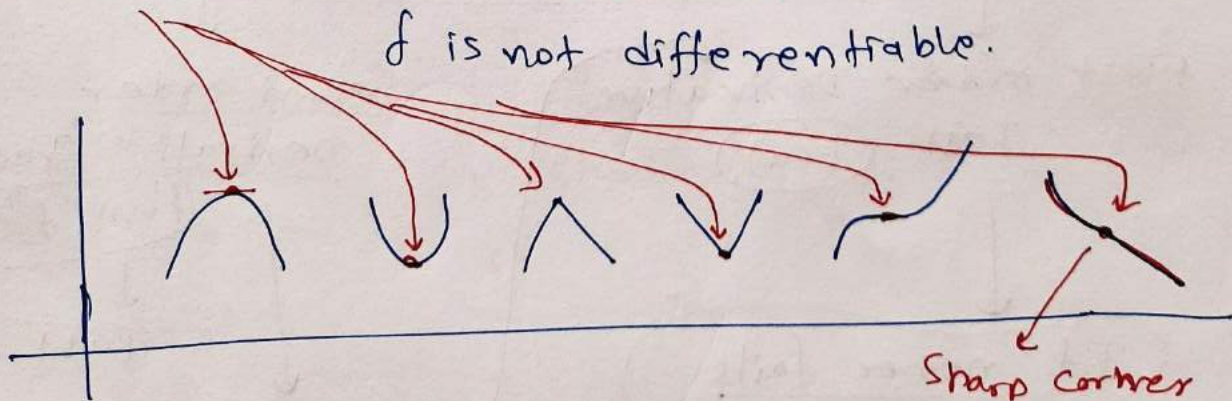
local minimum value
(relative minimum value) } → minimum in very close neighbourhood

local maximum value
(relative maximum value) } → maximum in very close neighbourhood

(local minima / local maxima) → local extrema

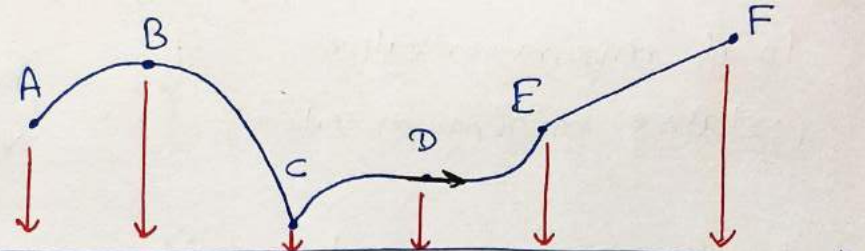


Critical point where $f'(x) = 0$ or f is not differentiable.



e.g.

Table of Tags

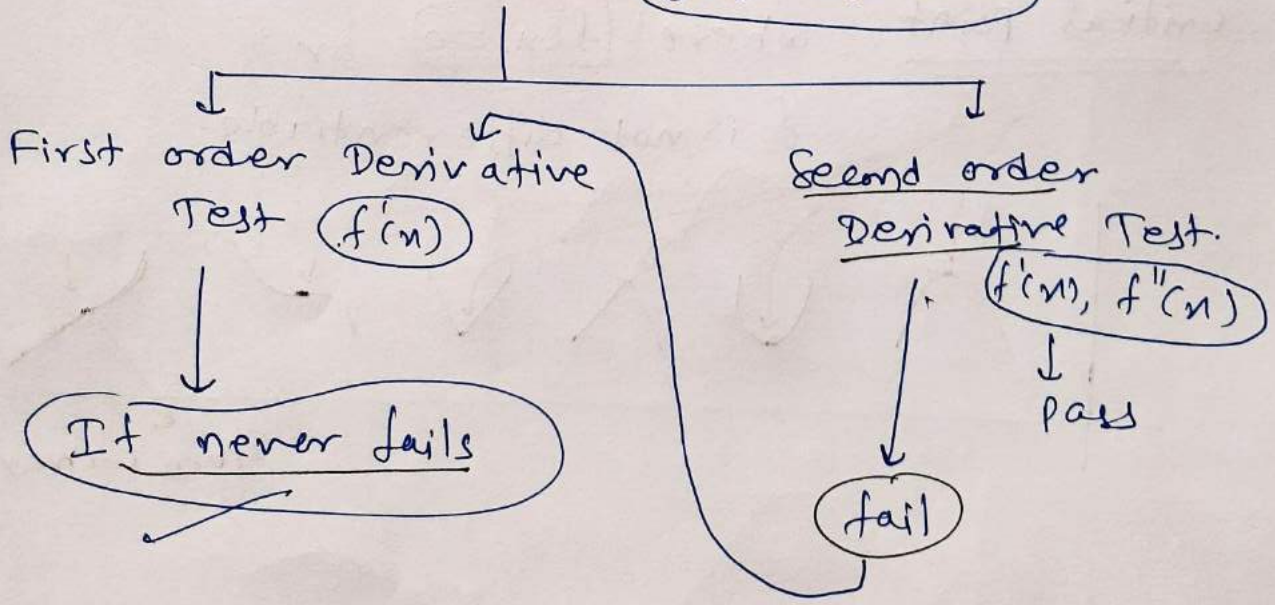


Extrema	→	✓	✓	✓	✗	✗	✓
↳ Minima	→	✓	✗	✓	✗	✗	✗
↳ Maxima	→	✗	✓	✗	✗	✗	✓
Turning Point	→	✗	✓	✓	✗	✗	✗
↳ Local minima	→	✗	✗	✓	✗	✗	✗
↳ Local maxima	→	✗	✓	✗	✗	✗	✗
Critical Point	→	✗	✓	✓	✓	✓	✗
↳ $f'(x)=0$	→	✗	✓	✗	✓	✗	✗
↳ f not diff.	→	✗	✗	✓	✗	✓	✗

Absolute minima (pointing to C)
 Point of inflection (circled, pointing to D)
 Absolute maxima (pointing to F)

How to find local minima / local maxima?

local extrema



First order Derivative Test to Find Local Extrema

$$f'(c) = 0$$

↓
Solve $x=c$

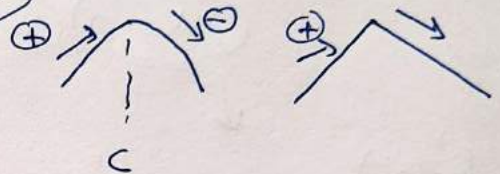
where f not differentiable

↓
 $x=c$

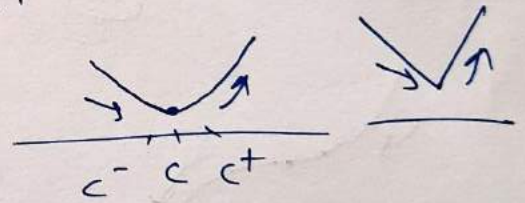
(III)

Let $x=c$ is the critical point

(I) if $f'(c^-) = \oplus$ & $f'(c^+) = \ominus$ then
Point of local maxima.

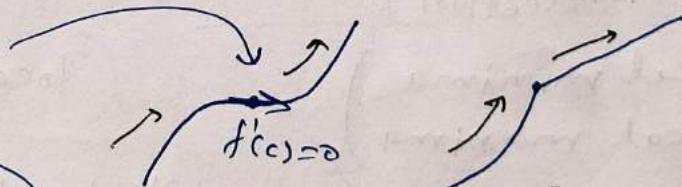


(II) if $f'(c^-) = \ominus$ & $f'(c^+) = \oplus$ then
(घटाना) (बढ़ाना)
Point of local minima.



(III) If $f'(c^-) = \oplus$ & $f'(c^+) = \oplus$
then point of inflection.

No local minima
No local maxima



(Diff.)

e.g. Find local minimum value and local maximum value of the function $f(x) = \frac{x^4}{4} - \frac{x^3}{3} + 7$.

Ans. 3rd order derivative test

$$f(x) = \frac{x^4}{4} - \frac{x^3}{3} + 7$$

$$f'(x) = x^3 - x^2 = 0 \quad (\text{Put}) \quad (\text{for critical point})$$

$$\Rightarrow x^2(x-1) = 0$$

$$x=0, x=1$$

$$f'(x) = x^2(x-1)$$

$$f'(0^-) \downarrow$$

$$\textcircled{+ \dots -}$$

$$= \textcircled{-}$$

$$f'(0^+) \downarrow$$

$$\textcircled{+ \dots -}$$

$$\downarrow$$

$$\textcircled{-}$$

$$f'(1^-) \downarrow$$

$$\textcircled{+ \dots -}$$

$$\downarrow$$

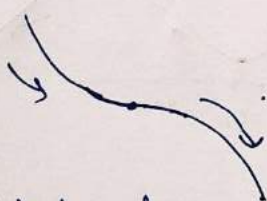
$$\textcircled{-}$$

$$f'(1^+) = \downarrow$$

$$\textcircled{+ \dots +}$$

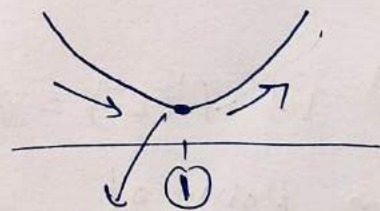
$$\downarrow$$

$$\textcircled{+}$$



(Point of inflection)

(No local minima)
(No local maxima)



(local minima)

$x=1$ = point of local minima.

No local maxima

$$\text{local minimum value} = f(1) = \frac{1^4}{4} - \frac{1^3}{3} + 7$$

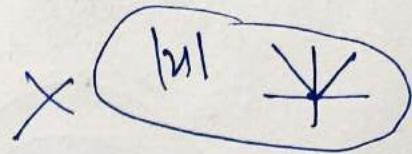
$$= \frac{1}{4} - \frac{1}{3} + 7$$

$$= \frac{3 - 4 + 84}{12} = \frac{83}{12}$$

Second Order Derivative Test

$$f''(x) = \frac{d^2y}{dx^2}$$

(only for twice differentiable functions)



$$f'(x) = 0$$

Critical points

- $x = \alpha$
- $x = \beta$
- $x = \gamma$

$$f''(x)$$

$$\rightarrow f''(\alpha) = \oplus \text{ve}$$

local minima

$$\rightarrow f''(\beta) = \ominus \text{ve}$$

local maxima

$$\rightarrow f''(\gamma) = 0$$

→ 2nd order derivative

Test

fail

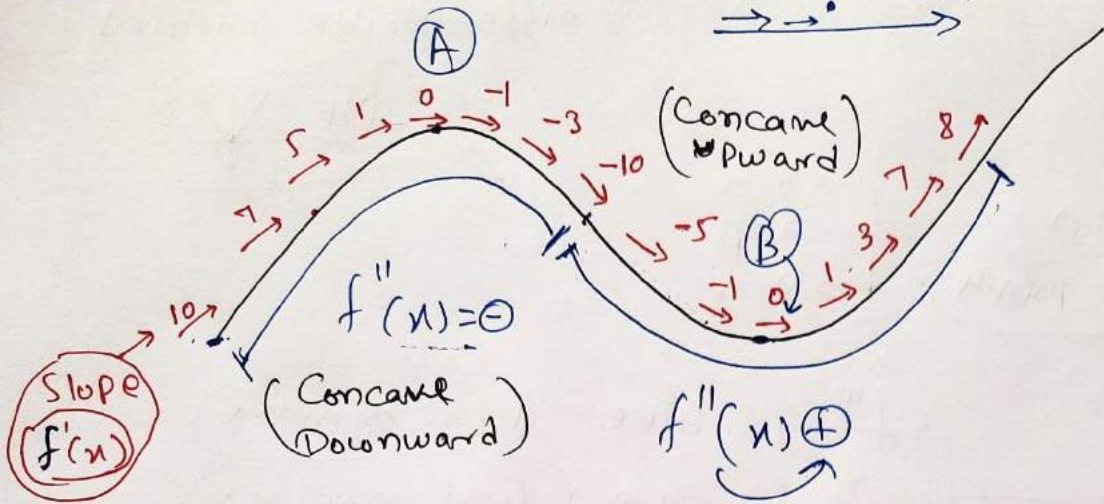
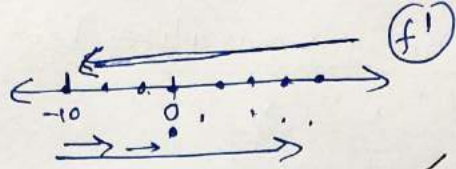
1st order derivative Test

$$f'(x^-)$$

$$f'(x^+)$$



Meaning of $f''(x)$



at A local maxima $f'(x) = 0$
 $f''(x) = -$

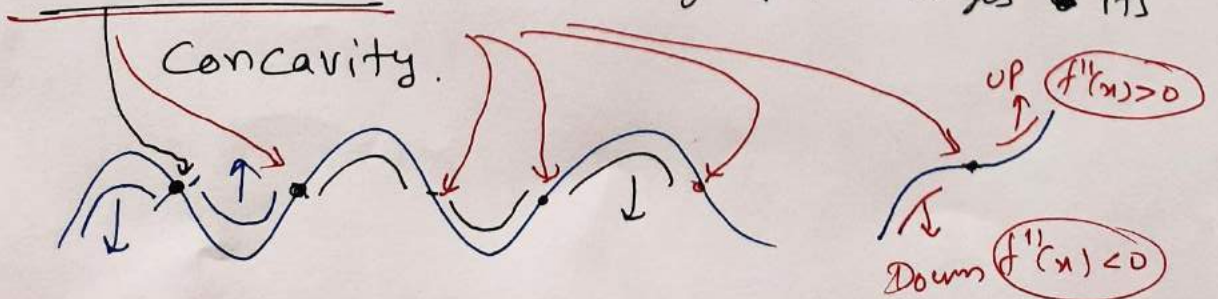
at B local minima $f'(x) = 0$
 $f''(x) = +$

Note

If $f''(x) > 0$, Concave upward Graph. $\curvearrowright \curvearrowright \curvearrowright$
 (उपर की ओर आकृत)

If $f''(x) < 0$, Concave downward graph $\curvearrowleft \curvearrowleft \curvearrowleft$
 (नीचे की ओर आकृत)

Point of inflection: where graph changes its



Q. Find local ~~max~~ maximum and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12.$$

Ans.

2nd order derivative test

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

$$f'(x) = 12x^3 + 12x^2 - 24x = 0 \quad (\text{For critical points})$$

$$f'(x) = 12x(x^2 + x - 2) = 0$$

$$12x(x-1)(x+2) = 0$$

$$x=0, x=1, x=-2$$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

$$f''(0) = -24 = \ominus \text{ve} \quad x=0 \text{ is local maxima}$$

$$f''(1) = 36 = \oplus \text{ve} \quad x=1 \text{ is local minima}$$

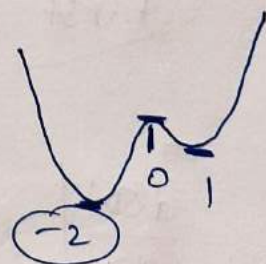
$$f''(-2) = 72 = \oplus \text{ve} \quad x=-2 \text{ is local minima}$$

$$\text{local maximum value} = f(0) = 12$$

$$\text{local minimum value} = f(1) = 7$$

$$\text{local minimum value} = f(-2) = -20$$

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$



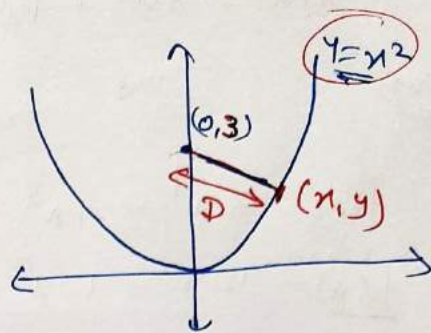
P.g. Find the shortest distance of the point

$(0, 3)$ from the parabola $y = x^2$.

Ans. y -axis

h

$y = x^2$ — (1)



Distance between

$$(0, 3) \text{ \& } (x, y) = D = \sqrt{(x-0)^2 + (y-3)^2}$$

$$\Rightarrow D = \sqrt{x^2 + (y-3)^2}$$

$x^2 = y$

$$\Rightarrow D = \sqrt{y + (y-3)^2}$$

by diff. w.r.t. y

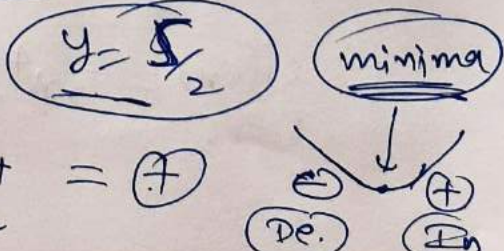
$$\Rightarrow \frac{d(D)}{dy} = \frac{1 \cdot (1 + 2(y-3))}{2\sqrt{y + (y-3)^2}}$$

$$\Rightarrow \frac{d(D)}{dy} = \frac{2y - 5}{2\sqrt{y + (y-3)^2}} = 0 \text{ (For Critical Point)}$$

First order Derivative Test.

$$\frac{d(D)}{dy} \Big|_{5/2^-} = (-)$$

$$\frac{d(D)}{dy} \Big|_{5/2^+} = (+)$$



$\therefore y = \frac{5}{2}$ is the point of local minima.

$$D = \sqrt{y + (y-3)^2}$$

Distance \rightarrow minimum \leftarrow Shortest Distance

$y = \frac{5}{2}$ put.

$$\begin{aligned} \text{Shortest Distance} &= \sqrt{\frac{5}{2} + \left(\frac{5}{2} - 3\right)^2} \\ &= \sqrt{\frac{5}{2} + \frac{1}{4}} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2} \end{aligned}$$

e.g. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given Cone is half of that of the cone. Constant

Ans. $CSA = 2\pi r h$ (Cylinder) \leftarrow Greatest CSA \leftarrow Cone \rightarrow Cylinder

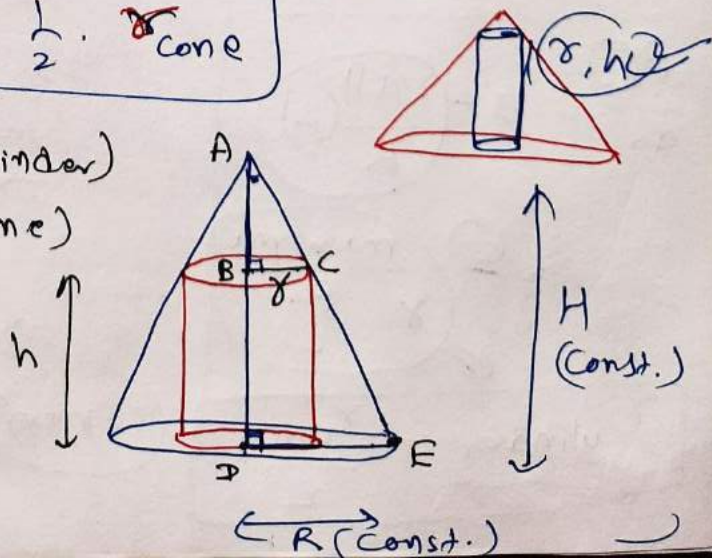
To prove: $r_{\text{cylinder}} = \frac{1}{2} \cdot r_{\text{cone}}$

Proof: $r, h \rightarrow$ variable (cylinder)
 $R, H \rightarrow$ constant (cone)

Rule of Similarity AA

$$\triangle ABC \sim \triangle ADE$$

$$\frac{AB}{AD} = \frac{BC}{DE} \Rightarrow \frac{H-h}{H} = \frac{r}{R}$$



$$\frac{H-h}{H} = \frac{r}{R}$$

$$\Rightarrow RH - R(h) = rH$$

$$\Rightarrow \underline{RH} - rH = R(h)$$

$$\Rightarrow \frac{H(R-r)}{R} = (h)$$

$$\text{CSA} = 'A' = 2\pi r(h)$$

(let)

$$\Rightarrow A = 2\pi r \left(\frac{H(R-r)}{R} \right)$$

$$\Rightarrow A = 2\pi rH - \frac{2\pi r^2 H}{R}$$

variable = r

by diff. w.r.t. 'r'

$$\frac{dA}{dr} = 2\pi H - \frac{2\pi H}{R}(2r) = 0 \quad (\text{for Critical Points})$$

2nd - order D.T.

$$\frac{d^2A}{dr^2} = 0 - \frac{2\pi H(2)}{R}$$

$$= \ominus \quad \underline{\text{maxima}}$$

$$r = \frac{R}{2}$$

$$\underline{\underline{r_{\text{cylinder}} = \frac{r_{\text{cone}}}{2}}}$$

$$\text{CSA of cylinder} = \underline{\underline{A = 2\pi rh}}$$



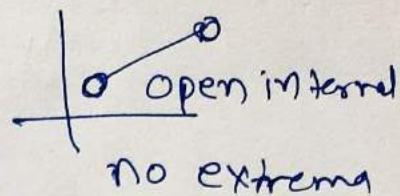
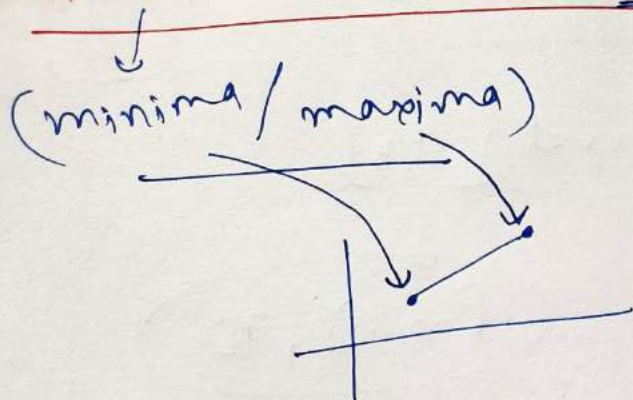
Optimize

Differentiation

$$\Rightarrow r = \frac{R}{2} \quad \text{Critical Point}$$

When

Extrema in the closed Interval.

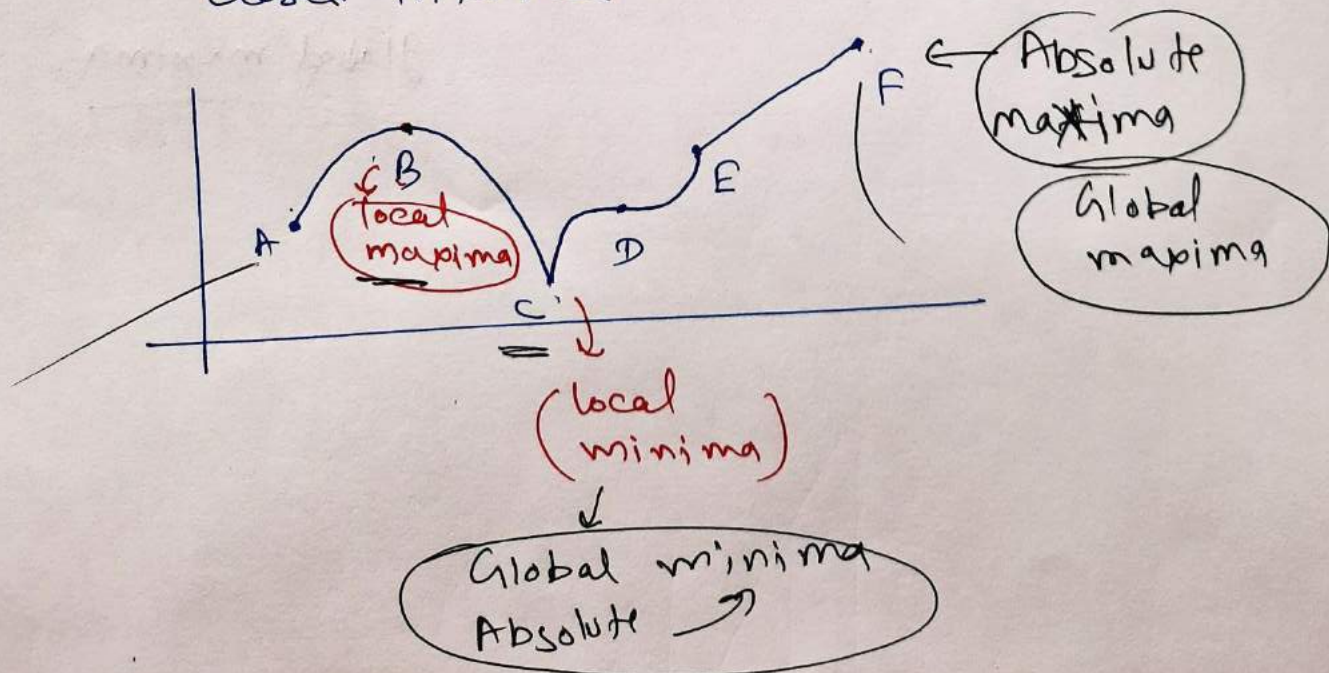


★ Absolute maxima (Global ~~max~~ maxima, Greatest value)

↓
Highest point of the curve
in that closed interval.

★ Absolute minima (Global minima, least value)

↓
lowest point of the curve in that
closed interval.



How to find absolute (global) minima/maxima?

$y = f(x)$,

interval $[a, b]$

local minima
local maxima

$f'(x) = 0$ \cup \cap

$x = \alpha$ $x = \beta$

अथवा x

$f(a), f(b), f(x)$

Find the value of func.

$f(a) =$
 $f(b) =$
 $f(\alpha) =$
 $f(\beta) =$

minimum among all these \downarrow
Global minima

maxima among all these \downarrow
Global maxima.

e.g. Find the absolute maximum and minimum of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Ans.

$$f'(x) = 6x^2 - 30x + 36$$

$$f'(x) = 6(x^2 - 5x + 6)$$

$$f'(x) = 6(x-2)(x-3) = 0 \text{ (For critical points)}$$

$$x = 2, 3 \in [1, 5]$$

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

किन्तु पल

$$x=1, f(1) = 2 - 15 + 36 + 1 = 24 \downarrow$$

$$x=5, f(5) = 2(125) - 15(25) + 36(5) + 1 = 56 \uparrow$$

बीच में

$$x=2, f(2) = 16 - 60 + 72 + 1 = 29$$

$$x=3, f(3) = 54 - 135 + 108 + 1 = 28$$

Global (Absolute) maximum value = 56 ($x=5$)

Absolute minimum value = 24 (at $x=1$)

e.g. Find absolute maximum and minimum values of a function f given by

$$\underline{f(x) = 12x^{4/3} - 6x^{1/3}}, \quad x \in [-1, 1].$$

Ans. $f'(x) = 12 \times \frac{4}{3} \cdot x^{1/3} - 6 \times \frac{1}{3} \cdot x^{1/3-1}$

$$\frac{1}{3} - 1 = -\frac{2}{3}$$

$$f'(x) = 16x^{1/3} - \frac{2}{x^{2/3}}$$

$$\frac{1}{3} + \frac{2}{3} = 1$$

$$f'(x) = \frac{16x - 2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}} = 0$$

for Critical Points

$$f'(x) = 0$$

Critical point

$$x = \frac{1}{8}$$

$$x = 0$$

$$f'(x) = \infty \quad (x=0)$$

Not Differentiable
Not Defined

$$x \in [-1, 1]$$

$$f(x) = 12x^{4/3} - 6x^{1/3}$$

कि-नारे पर

$$x = -1 \rightarrow f(-1) = 18$$

$$x = 1 \rightarrow f(1) = 6$$

Absolute maxima

बीचमें

$$x = 0 \rightarrow f(0) = 0$$

$$x = \frac{1}{8} \rightarrow f\left(\frac{1}{8}\right) = -\frac{9}{4}$$

Absolute minima

Absolute maximum value = 18 (at $x = -1$)

Absolute minimum value = $-\frac{9}{4}$ (at $x = \frac{1}{8}$)

Exercise-6.5

Chapter-6

Q.1 Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x-1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ (iv) $g(x) = x^3 + 1$

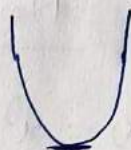
Ans. (i) $f(x) = (2x-1)^2 + 3, x \in \mathbb{R}$

(perfect square) ≥ 0

$\Rightarrow (2x-1)^2 \geq 0$

$\Rightarrow (2x-1)^2 + 3 \geq 3$

$\Rightarrow f(x) \geq 3$



minimum value of $f(x) = 3$

No maximum value

(ii) $f(x) = 9x^2 + 12x + 2$

$f(x) = \underbrace{(3x)^2 + 2 \cdot (3x)(2) + 2^2}_{a^2 + 2ab + b^2 = (a+b)^2} - 2^2$

$f(x) = (3x+2)^2 - 2$

we know that $(3x+2)^2 \geq 0$

$\Rightarrow (3x+2)^2 - 2 \geq -2$

$f(x) \geq -2$

minimum value of $f(x) = -2$

(no maximum value)

$$(iii) f(x) = -(x-1)^2 + 10$$

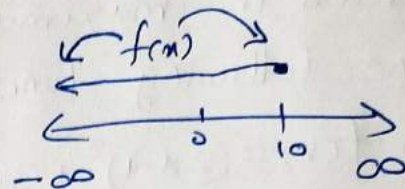
$$(x-1)^2 \geq 0$$

$$\Rightarrow -(x-1)^2 \leq 0$$

$$\Rightarrow \boxed{-(x-1)^2 + 10} \leq 10$$

$$f(x) \leq 10$$

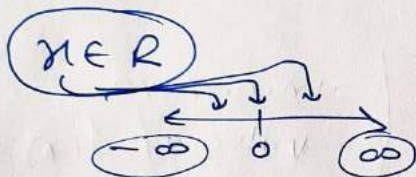
$$\begin{matrix} a > b \\ -a < -b \end{matrix}$$



maximum value of $f(x) = 10$

No minimum value

$$(iv) g(x) = x^3 + 1$$



$$x^3 \in \mathbb{R}$$

$$-\infty < x < \infty$$

$$\Rightarrow -\infty < x^3 < \infty$$

$$\Rightarrow -\infty < \boxed{x^3 + 1} < \infty$$

$$-\infty < g(x) < \infty$$

No maximum value \sim

No minimum value \sim

$$\begin{matrix} (-)^3 = - \\ (+)^3 = + \end{matrix}$$

$$\begin{matrix} (-)^2 = + \\ (+)^2 = + \end{matrix}$$

$$\begin{matrix} (-)^{\text{odd}} = \ominus \\ (+)^{\text{odd}} = \oplus \end{matrix} \quad \star$$

Q.2 Find the maximum value and minimum values, if any, of the following functions given by

(i) $f(x) = |x+2| - 1$

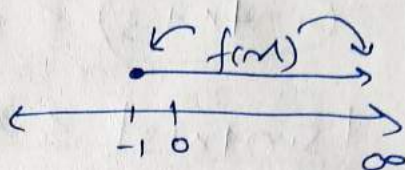
$(x)^2 \geq 0$

$|x+2| \geq 0$

$|x| \geq 0$

$|x+2| - 1 \geq -1$

$f(x) \geq -1$



minimum value of $f(x) = -1$

no maximum value

(ii) $f(x) = -|x+1| + 3$

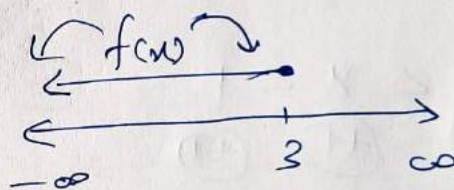
$|x+1| \geq 0$

$\Rightarrow -|x+1| \leq 0$

$\Rightarrow -|x+1| + 3 \leq 3$

$\Rightarrow f(x) \leq 3$

$|1| \geq 0$



maximum value of $f(x) = 3$

no minimum value.

(iii) $h(x) = \sin 2x + 5$

we know $-1 \leq \sin 2x \leq 1$
that $+5 \quad +5 \quad +5$

$-1 \leq \sin \theta \leq 1$
 $-1 \leq \cos \theta \leq 1$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$

$4 \leq h(x) \leq 6$

minimum value of $h(x) = 4$
maximum value of $h(x) = 6$

$$(iv) f(x) = |\sin 4x + 3|$$

$$\begin{array}{ccc} -1 & \leq & \sin 4x & \leq & 1 \\ +3 & & +3 & & +3 \end{array}$$

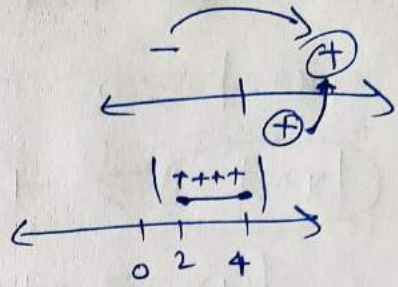
$$2 \leq \sin 4x + 3 \leq 4$$

$$2 \leq |\sin 4x + 3| \leq 4$$

$$2 \leq f(x) \leq 4$$

minimum value of $f(x) = 2$

maximum value of $f(x) = 4$



$$(v) h(x) = x+1, x \in (-1, 1)$$

open interval.

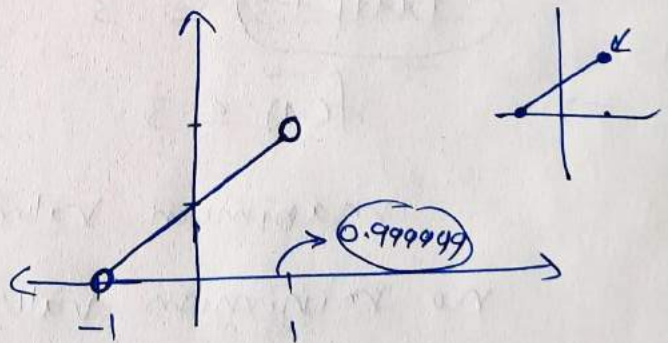
$$y = x+1$$

$$-1 < x < 1$$

$$\begin{array}{ccc} (+) & & (+) & & (+) \end{array}$$

$$0 < \underline{x+1} < 2$$

$$0 < h(x) < 2$$



No minimum value
No maximum value

} →

Q.3 Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and local minimum value.

(i) $f(x) = x^2$ local maxima \rightarrow Point of local maxima $x=c$

local maximum value $\rightarrow f(c)$

First order derivative test \rightarrow Never fails
 Second order derivative test

By first order derivative test

$f(x) = x^2$

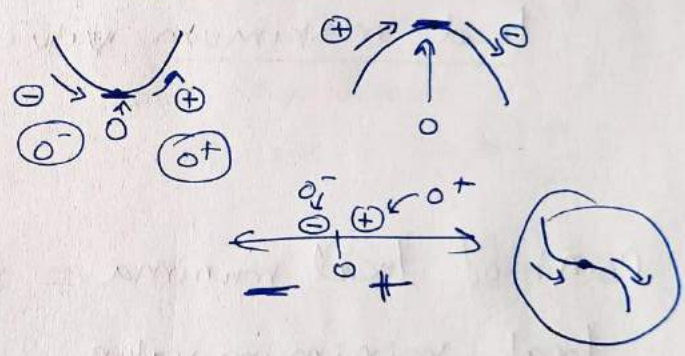
$f'(x) = 2x = 0$ (for critical points)

$x=0$

Slope = $f'(x) = 2x$

$f'(0^-) = 2(0^-) = -$

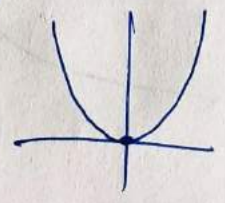
$f'(0^+) = 2(0^+) = +$



$\therefore x=0$ is the Point of local minima.

local minimum value of $f(x) = f(0) = 0^2 = 0$

$y = x^2$



x^2

(ii) $g(x) = \underline{x^3 - 3x}$ (second order derivative test)

$g'(x) = 3x^2 - 3 = 0$ (for critical points)

$g'(x) = 3(x^2 - 1) = 0$

$g'(x) = 3(x+1)(x-1) = 0$

$x = -1$ $x = 1$

$\Rightarrow g''(x) = 6x$ $\left\{ \begin{array}{l} \rightarrow g''(-1) = -6 = \ominus \text{ (local maxima)} \\ \rightarrow g''(1) = 6 = \oplus \text{ (local minima)} \end{array} \right.$

Point of local maxima = $x = -1$

$x^3 - 3x$

local maximum value of $g = g(-1) = (-1)^3 - 3(-1)$
 $= -1 + 3$
 $= 2$

Point of local minima = $x = 1$

local minimum value = $g(1) = 1 - 3 = -2$

(iii) $h(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

(First order derivative test)

$h'(x) = \cos x - \sin x = 0$ (for critical points)

$\Rightarrow \cos x = \sin x$

$\Rightarrow 1 = \tan x$

$x = \frac{\pi}{4}$

$f'(\frac{\pi}{4}^-)$

$f'(\frac{\pi}{4}^+)$

$$h(x) = \sin x + \cos x$$

Critical point

$$x = \frac{\pi}{4} \rightarrow 45^\circ$$

$$h'(x) = \cos x - \sin x$$

$$h'\left(\frac{\pi}{4}^-\right) = \cos \frac{\pi}{4} - \sin \frac{\pi}{4}$$

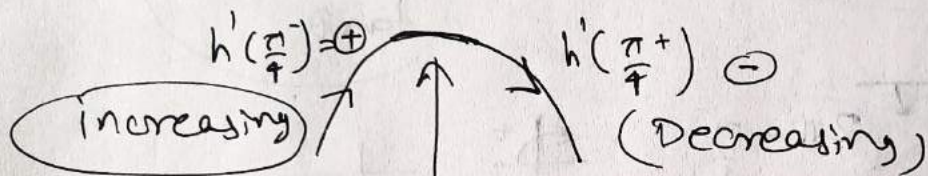
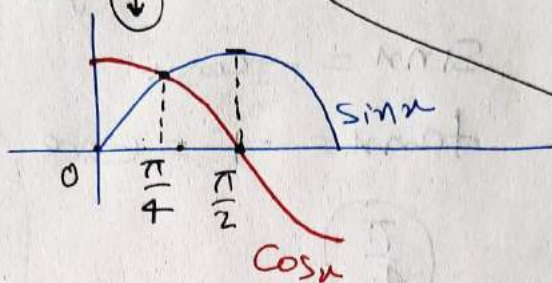
(↑) (↓)

$$= (+)$$

$$h'\left(\frac{\pi}{4}^+\right) = \cos \left(\frac{\pi}{4}^+\right) - \sin \left(\frac{\pi}{4}^+\right)$$

(↓) (↑)

$$= (-)$$



local maxima

$$x = \frac{\pi}{4}$$

local maximum

$$\text{value} = h\left(\frac{\pi}{4}\right)$$

$$2 = \frac{\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

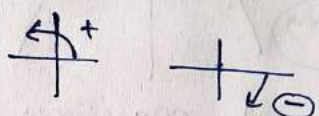
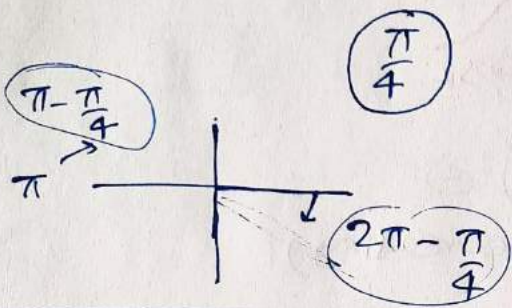
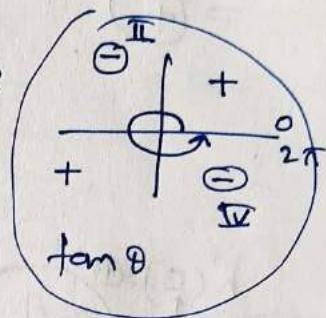
(iv) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

(we will do it by second order derivative test)

$f'(x) = \cos x + \sin x = 0$ (for critical points)

$\Rightarrow \sin x = -\cos x$

$\Rightarrow \tan x = -1 = \ominus ve$



$x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$

Critical points.

$f''(x) = -\sin x + \cos x$

$f''(\frac{3\pi}{4}) = -\sin(\frac{3\pi}{4}) + \cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} + (-\frac{1}{\sqrt{2}})$

$x = \frac{3\pi}{4}$ (local maxima) $= -\frac{2}{\sqrt{2}} = \ominus$

$f''(\frac{7\pi}{4}) = -\sin \frac{7\pi}{4} + \cos \frac{7\pi}{4} = +\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \oplus$

$x = \frac{7\pi}{4}$, (local minima)

$f(x) = \sin x - \cos x$

local maximum value $= f(\frac{3\pi}{4}) = \sin \frac{3\pi}{4} - \cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

local minimum value $= f(\frac{7\pi}{4}) = \sin \frac{7\pi}{4} - \cos \frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

$$\textcircled{V} f(x) = x^3 - 6x^2 + 9x + 15$$

(we will do it by 1st order derivative Test)

$$f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$$

$$x = 1, 3$$

(For Critical Points)

$$\boxed{x=1}$$

$$\boxed{x=3}$$

$$f'(1^-) = (-) \cdot (-) = (+)$$

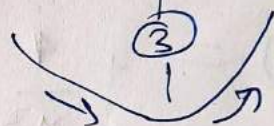
$$f'(1^+) = (+) \cdot (-) = (-)$$

$$f'(3^-) = (+) \cdot (-) = (-)$$

$$f'(3^+) = (+) \cdot (+) = (+)$$



$x=1$ point of local maxima



$x=3$ point of local minima.

$$\boxed{f(x) = x^3 - 6x^2 + 9x + 15}$$

local minimum value = $f(3)$ = $3^3 - 6(3)^2 + 9(3) + 15 = 15$

local maximum value = $f(1)$ = $1^3 - 6(1)^2 + 9(1) + 15 = 19$

$$\textcircled{\text{VI}} \quad g(x) = \frac{x}{2} + \frac{2}{x}, \quad x > 0$$

2^{nd} - order

$$g'(x) = \frac{1}{2} - \frac{2}{x^2}$$

$$g'(x) = \frac{x^2 - 4}{2x^2} = 0 \quad (\text{for critical points})$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x-2)(x+2) = 0$$

$$x = 2, \quad \cancel{-2} \quad -2 \neq 0$$

$$g''(x) = 0 - \frac{2}{x^3} \left(-\frac{2}{x^3} \right)$$

$$g''(x) = + \frac{4}{x^3}$$

$$g''(2) = \frac{4}{2^3} = \oplus \rightarrow x=2 \text{ point of}$$

local ~~max~~ minima)

local minimum value = $g(2) = \frac{2}{2} + \frac{2}{2} = 2$

$$g(x) = \frac{x}{2} + \frac{2}{x}$$

$$\begin{array}{l} x^{-2} \quad x^3 \\ \downarrow \quad \downarrow \\ -2 \cdot x^{-3} \\ \left(-\frac{2}{x^3} \right) \end{array}$$

VII

$$g(x) = \frac{1}{(x^2+2)}$$

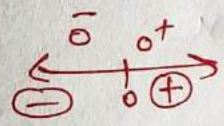
1st order

$$g'(x) = -\frac{1}{(x^2+2)^2} \times (2x)$$

$$g'(x) = \frac{-2x}{(x^2+2)^2} = 0 \text{ (for critical points)}$$

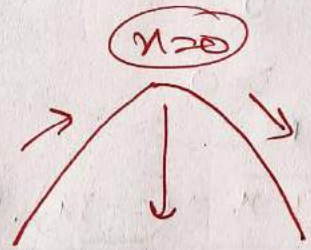
$$\Rightarrow -2x = 0$$

$$x = 0$$



$$g'(0^-) = -\frac{(-)}{(+)} = (+)$$

$$g'(0^+) = -\frac{(+)}{(+)} = (-)$$



Point of local maxima

$$x=0 \text{ for } g(x) = \frac{1}{x^2+2}$$

$$\text{local maximum value} = g(0) = \frac{1}{0^2+2} = \frac{1}{2}$$

VIII $f(x) = \frac{x \cdot \sqrt{1-x}}{2}$, $x > 0$
 (2nd order Den)

$$f'(x) = 1 \cdot \sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}} \quad (0-1)$$

$$f'(x) = \frac{\sqrt{1-x}}{1} - \frac{x}{2\sqrt{1-x}} = \frac{2(1-x) - x}{2\sqrt{1-x}}$$

$$f'(x) = \frac{2 - 3x}{2\sqrt{1-x}} = 0 \quad (\text{for critical points})$$

$$\Rightarrow 2 - 3x = 0$$

$$\Rightarrow x = \frac{2}{3} > 0$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - u \cdot v'}{v^2}$$

$$f''(x) = \frac{(-3) \cdot 2\sqrt{1-x} - (2-3x) \cdot \frac{d(2\sqrt{1-x})}{dx}}{4(1-x)}$$

Put $x = \frac{2}{3}$

$$f''\left(\frac{2}{3}\right) = \frac{-3 \cdot 2\sqrt{1-\frac{2}{3}} - 0 \cdot (\quad)}{4\left(1-\frac{2}{3}\right)} = \frac{\ominus}{\oplus} = \ominus$$

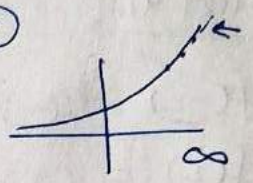
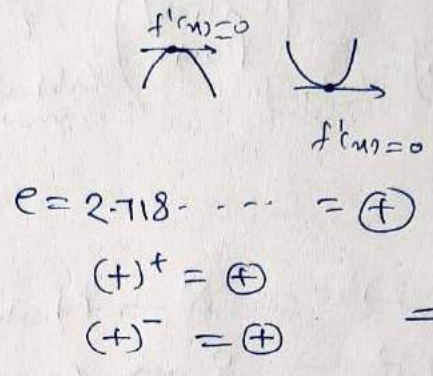
$x = \frac{2}{3}$ = Point of local maxima.

local maximum value = $f\left(\frac{2}{3}\right) = x\sqrt{1-x}$
 $= \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3\sqrt{3}}$

Exercise-6.5 Chapter - 6 (Maxima & minima)

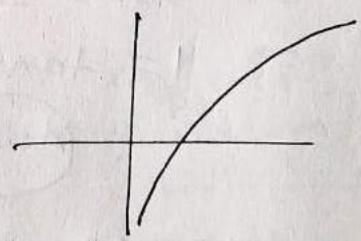
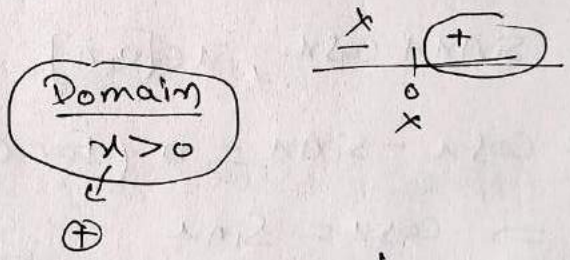
Q.4 Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$
 $f'(x) = e^x > 0$
 ↑
 (Slope)
 (Increasing)



$\therefore f(x) = e^x$ does not have any maxima or minima.

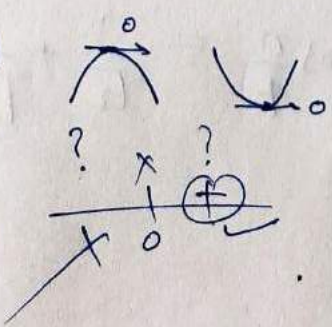
(ii) $g(x) = \log x$
 $g'(x) = \frac{1}{x}$
 $\frac{1}{x} = (+)$



$g'(x) = \frac{1}{x} > 0$ (Increasing)

No maxima, no minima

(iii) $h(x) = x^3 + x^2 + x + 1$
 $h'(x) = 3x^2 + 2x + 1 \neq 0$
 $D = b^2 - 4ac$
 $D = 4 - 4(3)(1)$
 $D = -8 < 0$



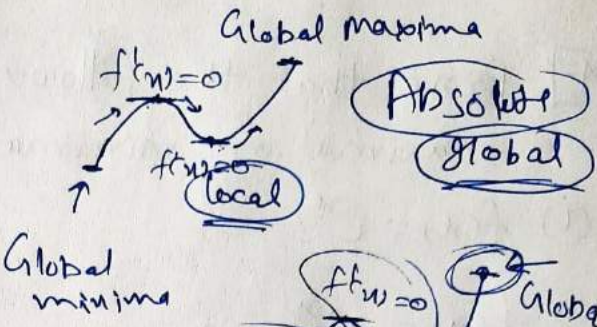
$h'(x) = 3x^2 + 2x + 1 > 0$
 (Slope) > 0 $(+)$ Increasing

No minima
 no maxima.

Q.5 Find the absolute maximum value and the absolute minimum value of the following function in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$

$f'(x) = 3x^2 = 0$ (for critical point)
 $\Rightarrow x=0 \in [-2, 2]$



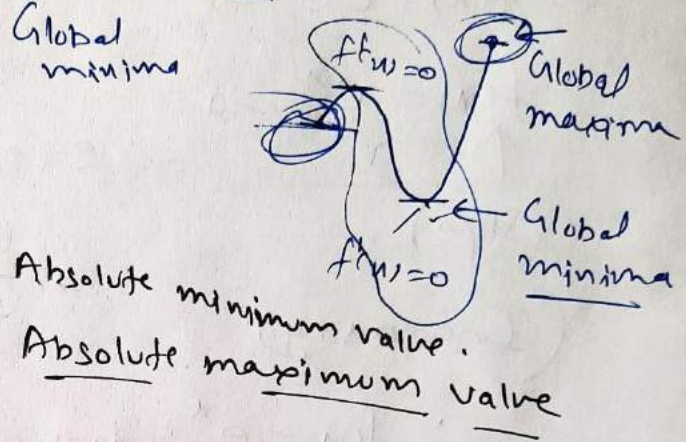
कि-व-रत-त-त:

$f(-2) = (-2)^3 = -8$

$f(2) = (2)^3 = 8$

की-व-रत-त:

$f(0) = (0)^3 = 0$



(ii) $f(x) = \sin x + \cos x, x \in [0, \pi]$

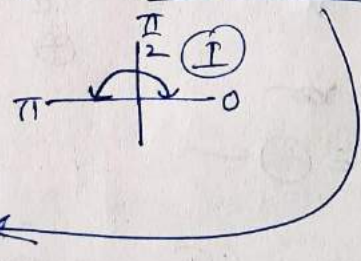
$f'(x) = \cos x - \sin x = 0$ (for critical points)

$\Rightarrow \cos x = \sin x$

$\Rightarrow 1 = \tan x$

(+)

$x = \frac{\pi}{4}$



$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$f(\pi) = \sin \pi + \cos \pi = 0 + (-1) = -1$ = absolute minimum value.

$f(\frac{\pi}{4}) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ = Absolute maximum value

$$(iii) f(x) = 4x - \frac{1}{2}x^2, \quad x \in \left[-2, \frac{9}{2}\right]$$

$$f'(x) = 4 - x = 0 \quad (\text{for Critical Points})$$

$$\Rightarrow \boxed{x=4} \in \left[-2, \frac{9}{2}\right]$$

$$f(4) = 4(4) - \frac{1}{2}(4)^2 = 8 \quad \leftarrow \text{Absolute maximum value.}$$

$$f(-2) = 4(-2) - \frac{1}{2}(-2)^2 = -10 \quad \leftarrow \text{Absolute minimum value}$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = \frac{18}{1} - \frac{81}{8} = \frac{144 - 81}{8}$$

$$= \frac{63}{8} \quad \leftarrow$$

$$\frac{64}{8}$$

$$(iv) f(x) = (x-1)^2 + 3, \quad x \in [-3, 1]$$

$$f'(x) = 2(x-1) = 0 \quad (\text{for Critical Points})$$

$$\Rightarrow \boxed{x=1}$$

$$f(-3) = (-3-1)^2 + 3 = 19 \quad \leftarrow \text{Absolute maximum value.}$$

$$f(1) = (1-1)^2 + 3 = 3 \quad \leftarrow \text{Absolute minimum value.}$$

Q6 Find the maximum profit that a company can make, if the profit function is given by

$$P(x) = 41 - 24x - 18x^2$$

Ans.

$$P(x) = 41 - 24x - 18x^2 \quad (\text{Profit function})$$

$$P'(x) = 0 - 24 - 36x = 0 \quad (\text{Put})$$

(For critical points)

$$\Rightarrow -12(2 + 3x) = 0$$

$$\Rightarrow x = -\frac{2}{3}$$

$$P''(x) = 0 - 36$$

$$P''(x) = -36$$

$$P''\left(-\frac{2}{3}\right) = -36 \rightarrow \text{Negative}$$

(local maxima)
at $x = -\frac{2}{3}$

$$\underline{\text{local maximum value}} = P\left(-\frac{2}{3}\right) = 41 - 24x - 18x^2$$

$$= 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$$

$$= 41 + 16 - 8$$

$$= 41 + 8 = 49 = \text{maximum profit}$$

Q.7 Find both the ^{Absolute.} maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

Ans.

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 \quad [0, 3]$$

$$f'(x) = 12x^3 - 24x^2 + 24x - 48$$

$$f'(x) = 12(x^3 - 2x^2 + 2x - 4)$$

$$= 12[x^2(x-2) + 2(x-2)]$$

$$f'(x) = 12[x^2 + 2] \cdot (x-2) = 0 \quad (\text{For critical points})$$

$$x^2 + 2 \neq 0$$

$$x - 2 = 0$$

$$x^2 \neq -2$$

$$x = 2$$

$$x = 2 \in [0, 3]$$

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$$

$$f(0) = 25 = \text{Absolute maximum value,}$$

$$f(2) = \cancel{48} - 64 + \cancel{48} - 48(2) + 25 = -39$$

$$f(3) = \underline{243} - \underline{216} + \underline{108} - \underline{144} + 25$$

$$= 27 - 11$$

$$= 16$$

Absolute minimum value.

Exercise-6.5

Chapter-6

minima & maxima

Q.8 At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

Ans.

$f(x) = \sin 2x, \quad x \in [0, 2\pi]$

$f'(x) = 2\cos 2x = 0$ (For Critical Points)

$\Rightarrow \cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \in [0, 4\pi]$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (Critical Point)

$f(x) = \sin 2x$

$f(0) = \sin(2 \times 0) = 0$

$f(\frac{\pi}{4}) = \sin(\frac{\pi}{2}) = 1 \rightarrow$

$f(\frac{3\pi}{4}) = \sin(\frac{3\pi}{2}) = -1$

$f(\frac{5\pi}{4}) = \sin(\frac{5\pi}{2}) = 1 \rightarrow$

$f(\frac{7\pi}{4}) = \sin(\frac{7\pi}{2}) = -1$

$f(2\pi) = \sin(4\pi) = 0$

$f(x) = 0$

Absolute maxima



local

$f'(x) = 0$

$x \in [0, 2\pi]$

$x \in [0, 4\pi]$

$\cos x = \cos \theta$

$x = 2n\pi \pm \theta$

maximum values

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Q.9) what is the maximum value of the function $\sin x + \cos x$?

Ans.

$$f(x) = \sin x + \cos x$$

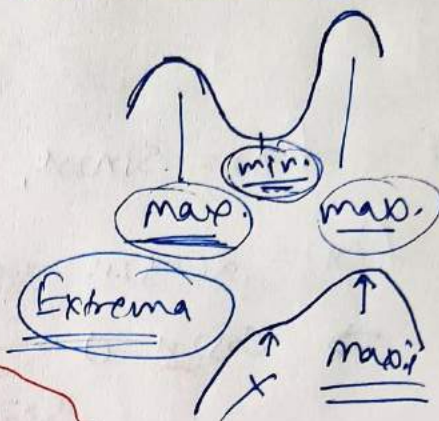
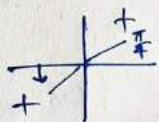
$$x \in \mathbb{R}$$

$$f'(x) = \cos x - \sin x = 0 \quad (\text{for critical points})$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow 1 = \tan x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$



Since minima and maxima occurs alternatively,

~~there~~ we have to take at least two critical points for complete possibility of maxima.

2nd order derivative Test

$$f''(x) = -\sin x - \cos x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \ominus ve.$$

(local maxima)

$$f''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \oplus ve$$

(local minima)

\therefore maximum value of $\sin x + \cos x$ at $x = \frac{\pi}{4}$

$$\rightarrow \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \leftarrow$$

Q.10 Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

maximum \rightarrow absolute maximum.

(किन्तु 92)

(अथवा 81)

$f'(x) = 0$

Critical Point.

Ans. $f(x) = 2x^3 - 24x + 107$

$f'(x) = 6x^2 - 24$

$f'(x) = 6(x^2 - 4) = 0$ (for critical points)

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$x = 2, -2$

$[1, 3]$

$[-3, -1]$

For maximum value in $[1, 3]$

$f(x) = 2x^3 - 24x + 107$

$f(1) = 2 - 24 + 107 = 85$

$f(3) = 54 - 72 + 107 = 89$ \leftarrow maximum value

$f(2) = 16 - 48 + 107 = 75$ in $[1, 3]$

For maximum value in $[-3, -1]$

$f(-3) = -54 + 72 + 107 = 125$

$f(-1) = -2 + 24 + 107 = 129$

$f(-2) = -16 + 48 + 107 = 139$ \leftarrow maximum value

in $[-3, -1]$

Q.11 It is given that $x=1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$.
Find the value of a .

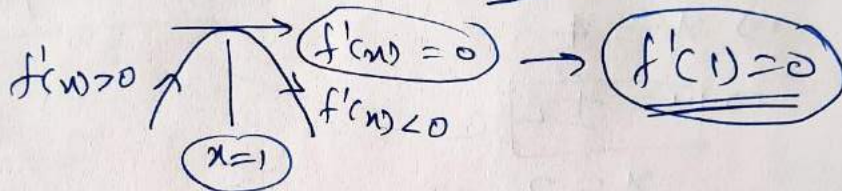
Ans.

$$f(x) = x^4 - 62x^2 + ax + 9$$

$$f'(x) = 4x^3 - 124x^2 + a = 0 \quad (\text{for critical point})$$

Satisfy $x=1$ \wedge $f'(x)=0$

at $x=1$, $f(x)$ gets a maxima.



$$f'(1) = 0$$

$$\Rightarrow 4(1)^3 - 124(1)^2 + a = 0$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow \boxed{a = 120} \quad \checkmark$$

$$\boxed{f(x) = x^4 - 62x^2 + 120x + 9}$$

$$x=1, x \in [0, 2]$$

$$f(0) = 9$$

$$f(1) = 1 - 62 + 120 + 9 = \boxed{68} \quad (\text{maximum})$$

$$f(2) = 16 - 248 + 240 + 9 = 17$$

$$\boxed{a = 120}$$

Q.12 Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

Ans. $f(x) = x + \sin 2x$

Absolute maxima
minima

$f'(x) = 1 + 2\cos 2x = 0$ (For Critical points)

$\Rightarrow \cos 2x = \left(-\frac{1}{2}\right) = \cos \frac{2\pi}{3}$

$x \in [0, 2\pi]$

$2x \in [0, 4\pi]$

Solution of $\cos x = \cos \theta$
 $\rightarrow x = 2n\pi \pm \theta$
 $n \in \mathbb{I}$

Solutions

$2x = 2n\pi \pm \frac{2\pi}{3} \quad \left\{ \dots, -2, -1, 0, 1, 2, 3, \dots \right\}$

$\Rightarrow x = n\pi \pm \frac{\pi}{3} \quad n \in \mathbb{I}$

$x \in [0, 2\pi]$

$n=0$

$x = \pm \frac{\pi}{3}$

$\rightarrow -\frac{\pi}{3} \notin [0, 2\pi]$

$\frac{\pi}{3}$

$n=1$

$x = \pi \pm \frac{\pi}{3}$

$\rightarrow \frac{2\pi}{3}$

$\rightarrow \frac{4\pi}{3}$

$n=2$

$x = 2\pi \pm \frac{\pi}{3}$

$\rightarrow \frac{5\pi}{3}$

$\rightarrow \frac{7\pi}{3}$

Critical points

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \quad x \in [0, 2\pi]$$

$$f(x) = x + \sin 2x$$

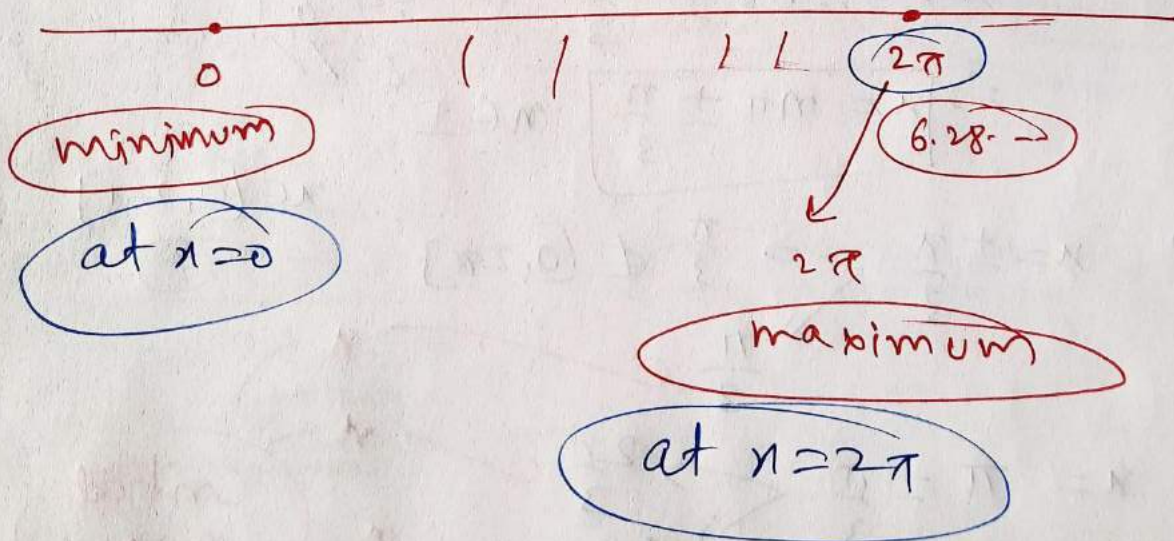
$$f(0) = 0 + 0 = 0, \quad f(2\pi) = 2\pi + \overset{0}{\sin 4\pi} = 2\pi$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \quad \times$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\left(\frac{4\pi}{3}\right) = \frac{2\pi}{3} + \sin\left(\pi + \frac{\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \quad \times$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \sin\left(2\pi + \frac{2\pi}{3}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \quad \times$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\left(\frac{10\pi}{3}\right) = \frac{5\pi}{3} - \frac{\sqrt{3}}{2} \quad \times$$



Exercise 6.5 Chapter 6

Q.13 Find two numbers whose sum is 24 and whose product is as large as possible.

Ans.

let x, y be two numbers

$$x + y = 24$$

$$y = 24 - x$$

let $x \times y \rightarrow$ maximum

$$f(x) = x(24 - x) \rightarrow \text{maximum}$$

$$f(x) = 24x - x^2$$

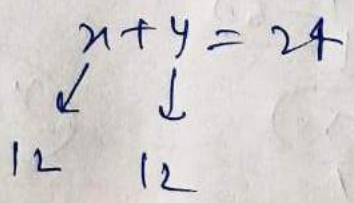
$$f'(x) = 24 - 2x = 0 \text{ (for critical points)}$$

$$\Rightarrow 24 = 2x$$

$$x = 12$$

$$f''(x) = 0 - 2 = -2$$

$$f''(12) = -2 = \ominus \text{ve (local maxima)}$$



maximum / minimum

3rd step $f(x)$ (y)

Variable

Differentiation $f'(x) = 0$

1st order Derivative 2^n

Q.14 Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

Ans. $x+y=60$
 $y = 60-x$
 $x = 60-y$

$xy^3 \rightarrow$ maximum.
 $\Rightarrow (60-y) \cdot y^3 \rightarrow$ maximum.
 $f(y)$

Let $f(y) = (60-y) \cdot y^3$
 $f(y) = 60y^3 - y^4 \rightarrow$ maximum

$f'(y) = 180y^2 - 4y^3 = 0$ (for critical points)

$\Rightarrow 4y^2 \cdot [45-y] = 0$
 $y=0, y=45$
 $x > 0$
 $y > 0$

Check

Second order Derivative Test (SODT)

$f''(y) = 360y - 12y^2 = 12y(30-y)$
 $y=45$

$f''(45) = 12 \times 45 \cdot (-15) = \ominus$ ve maxima

$y=45$

$x+y=60$
 $x=15, y=45$

$x \cdot y^3 \rightarrow$ maximum

Q.15 Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

Ans.

function,

$$x + y = 35$$

$$x^2 y^5 \rightarrow \text{maximum}$$

$$x = 35 - y \quad \text{--- (1)}$$

$$(35 - y)^2 \cdot y^5 \rightarrow \text{maximum.}$$

Let $f(y) = (35 - y)^2 \cdot y^5$
by diff. w.r.t. 'y'

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$\Rightarrow f'(y) = -2(35 - y) \cdot y^5 + (35 - y)^2 \cdot 5y^4$$

$$\Rightarrow f'(y) = (35 - y) \cdot y^4 \cdot [-2y + (35 - y) \cdot 5]$$

$$\Rightarrow f'(y) = (35 - y) \cdot y^4 \cdot (-7y + 35 \times 5) = 0$$

$$y = 35$$

$$y = 0$$

$$y = 25$$

(For critical points)

$$x = 0$$

$$y > 0$$

maxima

$$f'(y) = -7(35 - y) \cdot y^4 (y - 25)$$

$$f'(y) = -7(35 \cdot y^4 - y^5) \cdot (y - 25)$$

$$f''(y) = -7(140y^3 - 5y^4) \cdot (y - 25) - 7(35y^4 - y^5) \cdot 1$$

$$f''(25) = 0 - 7(\underline{35 \times 25^4} - \underline{25^5}) = \ominus$$

$$y=25, \quad x=10 \quad (\because x+y=35)$$

Q.16 Find two positive integers whose sum is 16 and the sum of whose cubes is minimum.

x, y
Given $x+y=16$
 $y=16-x$

x^3+y^3

$x^3+(16-x)^3$
f(x)

Let $f(x) = x^3 + (16-x)^3$

$$f(x) = x^3 + 4096 - x^3 + 3(16)(x^2) - 3(256)x$$

$$\Rightarrow f(x) = 3 \times 16 \times x^2 - 3(256)x + 4096$$

$$f'(x) = 3 \times 32 \times x - 3(256) = 0 \quad \left(\text{for Critical Points} \right)$$

$$3(32)(x-8) = 0$$

$x=8$

$$f''(x) = 3 \times 32$$

$$f''(8) = 3 \times 32 = \underline{\underline{+ve}} \quad \underline{\underline{\text{minima}}}$$

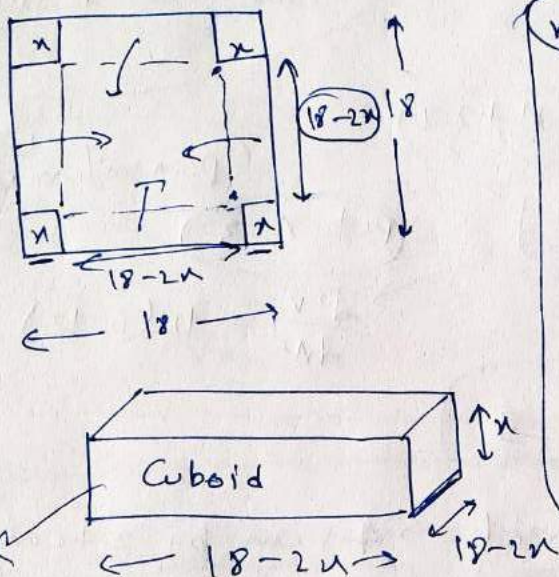
$x=8$
 $y=8$

$x+y=16$
8 8

Exercise 6.5 Chapter (6)

Q.17 A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of square to be cut off so that the volume of the box is the maximum possible.

Ans.



minimum / maximum
 \downarrow
 V, Area, length
 \downarrow
 1 variable
 Differentiate $f'(x) = 0$ $\left\{ \begin{array}{l} x=a \\ x=b \end{array} \right.$
 $f''(x)$
 $f''(a), f''(b)$
 \oplus minima \ominus maxima

Box Volume ~~is~~ maximum \uparrow

$x = ?$

Volume of Cuboid = $l b h$

$$V = (18-2x) \times (18-2x) \cdot x$$

$$V = (324 + 4x^2 - 72x) \cdot x$$

$$V = 4x^3 - 72x^2 + 324x$$

by differentiating w.r.t. 'x'

$$\frac{dV}{dx} = 12x^2 - 144x + 324 = 12(x^2 - 12x + 27)$$

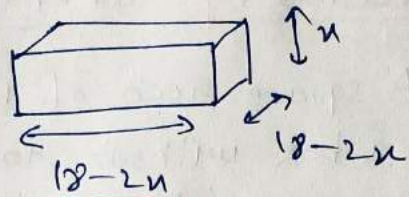
$$= 12(x-3)(x-9)$$

For critical points

put $\frac{dv}{dx} = 0$

$$\Rightarrow 12(x-3)(x-9) = 0$$

$$x = 3, x = 9$$



$x = 9$ $18 - 2x$
 $= 18 - 18$ Not possible
 Side = 0

local point of maximum

put $x = 3$

$$\frac{d^2v}{dx^2} = 12(6-12) = \underline{\underline{-ve}}$$

~~(ii)~~

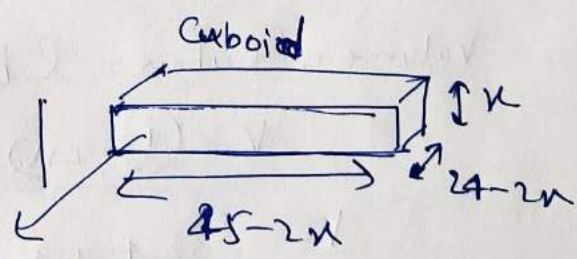
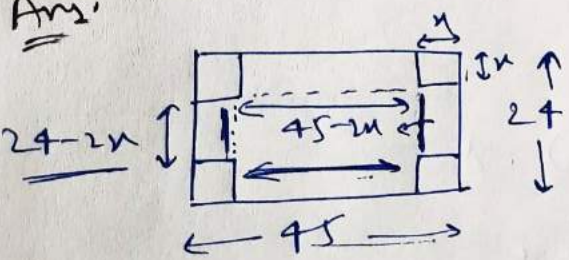
$$\frac{dv}{dx} = 12(x^2 - 12x + 27)$$

$$\frac{d^2v}{dx^2} = 12(2x - 12)$$

Side of square = $x = 3$

Q.18 A ~~reg~~ rectangular sheet \rightarrow 45 cm by 24 cm.
 Cut square from corners to make box.
 find side? volume maximum

Ans:



volume \rightarrow maximum

$$V = (45 - 2x)(24 - 2x)x$$

$$V = (1080 - 138x + 4x^2)x$$

$$V = 4x^3 - 138x^2 + 1080x$$

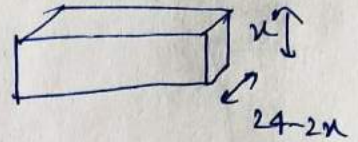
$$\frac{dv}{dx} = 12x^2 - 276x + 1080 = 0 \quad (\text{for critical points})$$

$$\Rightarrow 12(x^2 - 23x + 90) = 0$$

$$\Rightarrow 12(x^2 - 18x - 5x + 90) = 0$$

$$\Rightarrow 12(x-18)(x-5) = 0$$

$$x = 18, 5$$



$$x = 18$$

Side $24 - 36 = -12$

$$\frac{d^2v}{dx^2} = 24x - 276$$

$$x = 8$$

$$= 120 - 276 = -156$$

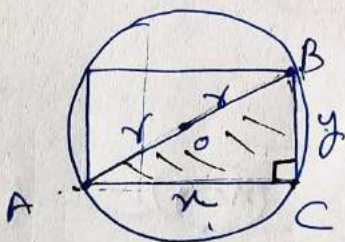
Max

Maxima

Side of Square = $x = 5$

[Q.19] Show that of all the rectangles inscribed in a given fixed circle, the square has maximum area.

Ans.



radius = fixed = r (let)
(Constant)

$A \rightarrow \text{max.}$
 $A^2 \rightarrow \text{max.}$

AB \rightarrow Rectangle's Diagonal

AB \rightarrow Circle's Diameter

$$AB = 2r = \text{Constant}$$

By PGT; $x^2 + y^2 = (2r)^2$

$$y = \sqrt{4r^2 - x^2}$$

Area of rectangle

$$A = x \times y$$

$$A = x \times \sqrt{4r^2 - x^2}$$

Square

$$A^2 = x^2 (4r^2 - x^2)$$

$$A^2 = 4r^2 x^2 - x^4$$

$$A^2 = 4r^2x^2 - x^4$$

by diff. w.r.t. x

$A \rightarrow \text{max.}$
 $A^2 \rightarrow \text{max.}$

$$\Rightarrow \frac{d(A^2)}{dx} = 8r^2x - 4x^3 = 0 \quad (\text{For critical points})$$

$$\Rightarrow \frac{4x(2r^2 - x^2)}{4} = 0$$

$$\Rightarrow x=0, \quad 2r^2 = x^2$$

$$x = \pm \sqrt{2}r$$

$x = +\sqrt{2}r$

$$\frac{d^2(A^2)}{dx^2} = 8r^2 - 12x^2$$

$$= 8r^2 - 12(2r^2)$$

$$= 8r^2 - 24r^2 = -16r^2 \quad (\text{maxima})$$

$x = \sqrt{2}r$

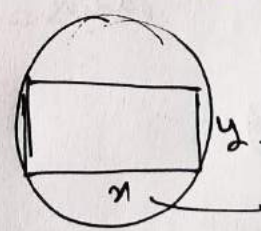
$$y = \sqrt{4r^2 - x^2}$$

$$y = \sqrt{4r^2 - 2r^2}$$

$$y = \sqrt{2r^2}$$

$y = \sqrt{2}r$

$x = y = \sqrt{2}r$



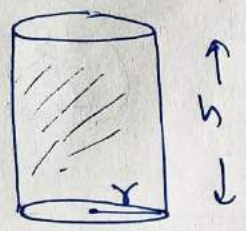
$x = y$

Square

area maximum

Q.20 Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of base.

Ans.



Given Surface = ^{total} Surface area = Fixed Constant

$$S = 2\pi r h + 2\pi r^2 \quad (1)$$

$r, h \rightarrow$ variable

maximum $V = \pi r^2 h \quad (2)$

$\frac{dV}{dr}$

By eqⁿ (1)

$$\frac{S - 2\pi r^2}{2\pi r} = h$$

By eqⁿ (2): $V = \pi r^2 h$

$$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r} \right)$$

$$V = \frac{Sr}{2} - \pi r^3$$

by diff. w.r.t 'r'

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0 \quad (\text{For C.P.})$$

$$\Rightarrow \frac{S}{2} = 3\pi r^2$$

$$S = 6\pi r^2 \rightarrow r = \sqrt{\frac{S}{6\pi}}$$

$$S = 2\pi r h + 2\pi r^2$$

$$S = 6\pi r^2$$

$$6\pi r^2 = 2\pi r h + 2\pi r^2$$

$$\Rightarrow 4\pi r^2 = 2\pi r h$$

$$2r = h$$

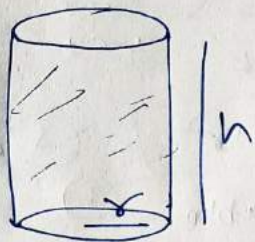
Diameter \rightarrow height

$$\frac{d^2V}{dr^2} = 0 - 6\pi$$

$= \ominus ve$ (maxima)

(Q. 2) • Of all the closed cylindrical cans, of a given volume of 100 cm^3 , find the ~~dimenstio~~ dimensions of the can which has the minimum surface area?

Ans.



$$V = 100 = \pi r^2 h$$

$$\left(\frac{dS}{dr} \right)$$

$$h = \frac{100}{\pi r^2}$$

$$S = \text{total surface area} = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi r^2 + 2\pi r \left(\frac{100}{\pi r^2} \right) = 2\pi r^2 + \frac{200}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2} = 0 \quad (\text{for C.P.})$$

$$\Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow 4\pi r^3 = 200$$

$$\Rightarrow r^3 = \left(\frac{200}{4\pi} \right) \Rightarrow r = \left(\frac{200}{4\pi} \right)^{\frac{1}{3}}$$

$$\frac{d^2 S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$= 4\pi + \frac{400}{\left(\frac{200}{4\pi} \right)}$$

= (+)ve

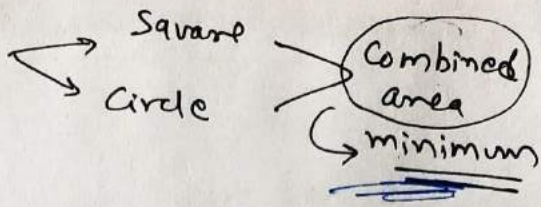
minima

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi \left(\frac{50}{\pi} \right)^{\frac{2}{3}}}$$

$$h = \frac{2 \times 50^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} \times \frac{\pi^{\frac{2}{3}}}{50^{\frac{2}{3}}} = 2 \frac{50^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$$

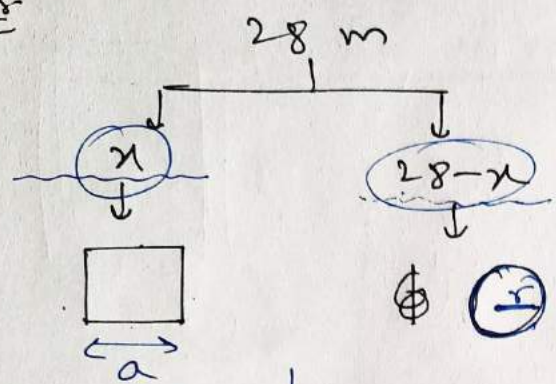
$$= 2 \left(\frac{50}{\pi} \right)^{\frac{1}{3}}$$

Q.22 length of wire = 28 m



length of two pieces = ?

Ans



Perimeter of Square = x

Circumference of circle = $28-x$

Combined area = A
= area of Square + area of circle.
 $= a^2 + \pi r^2$

$$A = a^2 + \pi r^2$$

$$\Rightarrow 4a = x$$

$$2\pi r = (28-x)$$

$$A = \left(\frac{x}{4}\right)^2 + \pi \left(\frac{28-x}{2\pi}\right)^2$$

$$a = \frac{x}{4}$$

$$r = \frac{28-x}{2\pi}$$

$$A = \frac{x^2}{16} + \frac{(28-x)^2}{4\pi}$$

minimum

C.P. $0 = \frac{dA}{dx} = \frac{2x}{16} - \frac{2(28-x)}{4\pi}$

$$\frac{d^2A}{dx^2} = \frac{2}{16} + \frac{1}{4\pi}$$

$$\Rightarrow \frac{2x}{16} = \frac{2(28-x)}{4\pi}$$

= (+)

$$\Rightarrow x\pi = 112 - 4x$$

minima

$$\Rightarrow x(\pi + 4) = 112$$

$$28-x = 28 - \frac{112}{\pi+4}$$

$$\Rightarrow x = \frac{112}{\pi+4}$$

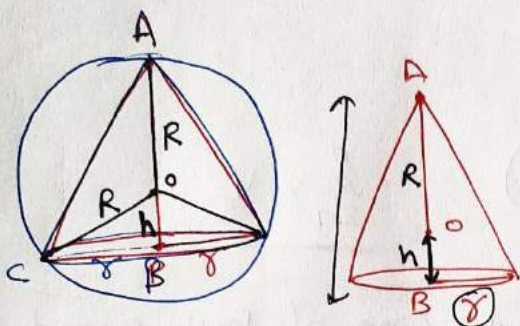
$$= \frac{28\pi + 112 - 112}{\pi+4} = \frac{28\pi}{\pi+4}$$

Exercise 6.5

Chapter 6

Q.23 Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

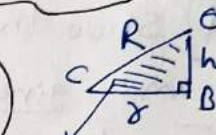
Ans.



Maximum & minimum
Prove → V, A, I
 Diff. → 1 variable

Radius of sphere = R
 Radius of cone = r
 height of cone = $R+h$

(Constants = R)
 Variable = r, h



$r^2 + h^2 = R^2$

Volume of Cone → maxima

$V_c = \frac{1}{3} \pi r^2 (R+h)$

$V_c = \frac{1}{3} \pi (R^2 - h^2) (R+h)$

$V_c = \frac{1}{3} \pi (R^3 + R^2h - Rh^2 - h^3)$

Diff. w.r.t. 'h'

$\frac{d(V_c)}{dh} = \frac{\pi}{3} (0 + R^2 - 2Rh - 3h^2) = 0 =$ (for Critical Points)

$\frac{d^2(V_c)}{dh^2} = \frac{\pi}{3} (-2R - 6h)$
 ↓
 one (maxima)

$\Rightarrow R^2 - 2Rh - 3h^2 = 0$
 $\Rightarrow R^2 - 3Rh + Rh - 3h^2 = 0$
 $\Rightarrow R(R-3h) + h(R-3h) = 0$
 $(R+h) \cdot (R-3h) = 0$
 $\Rightarrow h = \frac{R}{3}$

To Prove,
 $V_c = \frac{8}{27} \cdot V_s$
 \downarrow
 $\frac{4}{3} \pi R^3$

$$V_c = \frac{1}{3} \pi (R^2 - h^2)(R+h)$$

(Put $h = \frac{R}{3}$)

$$V_c = \frac{1}{3} \pi \left(R^2 - \frac{R^2}{9} \right) \left(R + \frac{R}{3} \right)$$

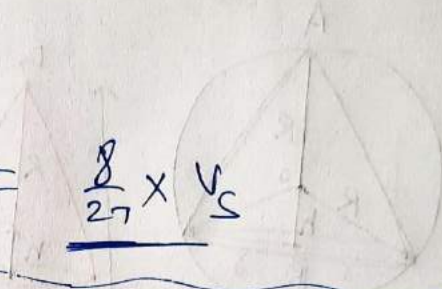
$$V_c = \frac{1}{3} \pi \left(\frac{8R^2}{9} \right) \left(\frac{4R}{3} \right)$$

$$V_c = \frac{8}{27} \left(\frac{4}{3} \pi R^3 \right) = \frac{8}{27} \times V_s$$

To Prove

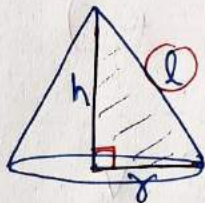
$$V_c = \frac{8}{27} V_s$$

$$\frac{4}{3} \pi R^3$$



[Q.24] Show that the right circular cone of least curved surface and given volume has ~~at~~ an altitude equal to $\sqrt{2}$ times the radius of Base.

Ans.



minima $CSA = A = \pi r l$

Volume = $V =$ given (Fix)

$$V = \frac{1}{3} \pi r^2 h$$

$$h = \frac{3V}{\pi r^2}$$

To Prove

$$h = \sqrt{2} r$$

(Height = h)

$$l^2 = h^2 + r^2$$

$$l = \sqrt{h^2 + r^2}$$

minima $A = \pi r l$

$$A = \pi r \sqrt{h^2 + r^2}$$

$$A = \pi r \sqrt{\frac{9V^2}{\pi^2 r^4} + r^2}$$

$$A^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2 \right)$$

$$A^2 = \frac{9V^2}{r^2} + \pi^2 r^4$$

$A \downarrow$
 $A^2 \downarrow$

by Diff. w.r.t. r

$$\frac{d(A^2)}{dr} = \frac{-18V^2}{r^3} + 4\pi^2 r^3 = 0$$

(for C.P.)

$$\Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3}$$

$$\Rightarrow 2\pi^2 r^6 = 9V^2$$

$$\Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

$$\frac{d(A^2)}{dr} = -\frac{18v^2}{r^3} + 4\pi^2 r^3$$

again by diff. w.r.t. 'r'

$$\Rightarrow \frac{d^2(A^2)}{dr^2} = +\frac{54v^2}{r^4} + 12\pi^2 r^2$$

$$\text{put } r^6 = \frac{9v^2}{2\pi^2}$$

$$= \oplus \text{ve} \quad \text{minima}$$

$$A^2 \downarrow$$

$$A \downarrow$$

we got $r^6 = \frac{9v^2}{2\pi^2}$

$$v = \frac{1}{3}\pi r^2 h$$

put

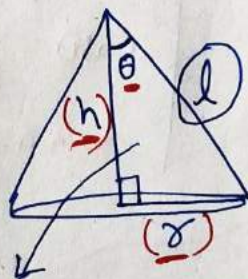
$$\Rightarrow r^6 = \frac{9}{2\pi^2} \left(\frac{1}{3}\pi r^2 \cdot h \right)^2$$

$$\Rightarrow 2r^2 = h^2$$

$$\Rightarrow \boxed{\sqrt{2} \cdot r = h}$$

Q. 25 Show that the semi vertical angle of the cone of the maximum volume ~~and~~ and of given slant height is $\tan^{-1} \sqrt{2}$.

Ans



To Prove

$$\theta = \tan^{-1} \sqrt{2}$$

Proof

$$V = \frac{1}{3}\pi r^2 h$$

(Slant height = l)
(Constant)

$$V = \frac{1}{3}\pi l^2 \sin^2 \theta \cdot l \cos \theta$$

$$V = \frac{1}{3}\pi l^3 \cdot \sin^2 \theta \cdot \cos \theta$$

$$\cos \theta = \frac{h}{l} \Rightarrow \boxed{h = l \cos \theta}$$

$$\sin \theta = \frac{r}{l} \Rightarrow \boxed{r = l \sin \theta}$$

$$V = \frac{1}{3} \pi l^3 \sin^2 \theta \cdot \cos \theta$$

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

by differentiating with respect to ' θ '

$$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3} \pi l^3 [2 \sin \theta \cdot \cos \theta \cdot \cos \theta - \sin^2 \theta \cdot \sin \theta]$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\frac{dV}{d\theta} = \frac{1}{3} \pi l^3 [2 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta]$$

$$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3} \pi l^3 (2 \sin \theta - 3 \sin^3 \theta) = 0$$

(For Critical Points)

$$\Rightarrow \sin \theta (2 - 3 \sin^2 \theta) = 0$$

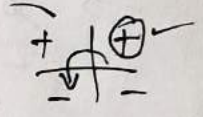
$$\frac{\sin \theta = 0}{\theta = 0, \pi}$$

$$\sin^2 \theta = \frac{2}{3}$$

$$\sin \theta = \pm \sqrt{\frac{2}{3}}$$

$$\sin \theta = +\sqrt{\frac{2}{3}}$$

$$\sin \theta = -\sqrt{\frac{2}{3}}$$



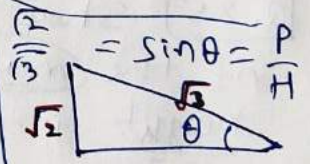
S.O.D.T.

$$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{1}{3} \pi l^3 (2 \cos \theta - 9 \sin^2 \theta \cdot \cos \theta)$$

$$= \frac{1}{3} \pi l^3 \left(\frac{2}{3} - 9 \cdot \left(\frac{2}{3}\right) \cdot \frac{1}{3} \right)$$

↓ - ↑

= 0ve **maxima**



$$\left. \begin{aligned} \cos \theta &= \frac{1}{3} \\ \tan \theta &= \sqrt{2} \end{aligned} \right\}$$

θ → variable

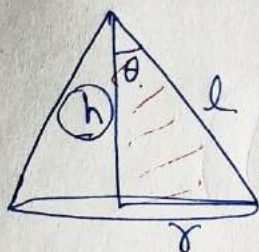
$$\sin \theta = \sqrt{\frac{2}{3}}$$

$$\tan \theta = \sqrt{2}$$

$$\theta = \tan^{-1} \sqrt{2}$$

Q.26 Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Ans.



To Prove $\theta = \sin^{-1}\left(\frac{1}{3}\right)$

Surface area = TSA = $A = \pi r l + \pi r^2$
 given (Constant)

$l = \sqrt{h^2 + r^2}$

Variables = θ, h, r, l, v

$h = \sqrt{l^2 - r^2}$

$V = \frac{1}{3} \pi r^2 h \leftarrow$ maximum (Diff.)

$V = \frac{1}{3} \pi r^2 \sqrt{l^2 - r^2}$

$A = \pi r l + \pi r^2$

$\left(\frac{A - \pi r^2}{\pi r}\right) = l$

$V = \frac{1}{3} \pi r^2 \sqrt{\left(\frac{A - \pi r^2}{\pi r}\right)^2 - r^2}$

$\Rightarrow V^2 = \frac{1}{9} \pi^2 r^4 \left[\frac{A^2 + \pi^2 r^4 - 2A\pi r^2}{\pi^2 r^2} - r^2 \right]$

$\Rightarrow V^2 = \frac{1}{9} [A^2 r^2 + \pi^2 r^6 - 2A\pi r^4 - \pi^2 r^6]$

$\Rightarrow V^2 = \frac{1}{9} [A^2 r^2 - 2A\pi r^4]$

$V \uparrow \Rightarrow V^2 \uparrow$ maximum

$\Rightarrow \frac{d(V^2)}{dr} = \frac{1}{9} (2A^2 r - 8A\pi r^3) = 0$ (For Critical Point)

$\left| \frac{1}{9} \cdot 2A \cdot r \cdot (A - 4\pi r^2) = 0 \right.$

$A = 4\pi r^2$ $\left(r^2 = \frac{A}{4\pi} \right)$

$$\frac{d(v^2)}{dr} = \frac{1}{9} (2A^2r - 8\pi Ar^3)$$

again by differentiating w.r.t. r

$$r = \frac{A}{4\pi}$$

$$\Rightarrow \frac{d^2(v^2)}{dr^2} = \frac{1}{9} (2A^2 - 24\pi Ar^2)$$

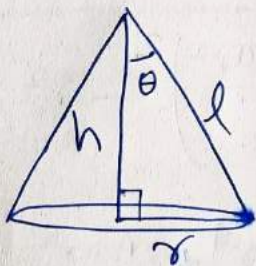
$$= \frac{1}{9} (2A^2 - \cancel{24} \pi A \left(\frac{A}{4\pi}\right))$$

$$= \frac{1}{9} (2A^2 - 6A^2) = \underline{\underline{-ve}}$$

$\ominus ve$

maxima

$$v^2 \uparrow = v \uparrow$$



$$\theta = \sin^{-1}\left(\frac{1}{3}\right)$$

To prove

$$\sin \theta = \frac{r}{l}$$

We got $A = 4\pi r^2$

TSA

$$\Rightarrow \pi r l + \pi r^2 = 4\pi r^2$$

$$\Rightarrow \cancel{\pi} r l = 3\cancel{\pi} r^2$$

$$\Rightarrow l = 3r$$

$$\Rightarrow \boxed{\frac{1}{3} = \frac{r}{l}}$$

$$\sin \theta = \frac{r}{l}$$

$$\sin \theta = \frac{1}{3}$$

$$\theta = \sin^{-1}\left(\frac{1}{3}\right)$$

Exercise 6.5

Chapter 6

Q27, Q28, Q29

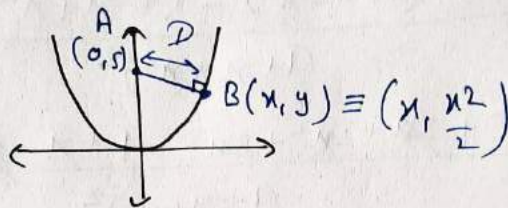
Q.27 The point on the curve $x^2 = 2y$ which is nearest to the point $(0, 5)$ is -

- (A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) $(0, 0)$ (D) $(2, 2)$

Ans.

$$2y = x^2$$

$$y = \frac{x^2}{2}$$



nearest
↓
(A) (D)
(AB = minimum)

$$\text{Distance b/w A \& B} = D = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2}$$

$$\Rightarrow D^2 = x^2 + \frac{x^4}{4} + 25 - 5x^2$$

by Diff.

$$\Rightarrow \frac{d(D^2)}{dx} = 2x + x^3 - 10x$$

$$= x^3 - 8x = 0 \quad (\text{For Critical Points})$$

$$\Rightarrow x(x^2 - 8) = 0$$

$$\underline{x=0}, x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$$

$$x = 2\sqrt{2}$$

$$x = -2\sqrt{2}$$

(Not in Options)

$$\frac{d^2(D^2)}{dx^2} = 3x^2 - 8$$

$$x=0 \quad \frac{d^2(D^2)}{dx^2} = -8 = \ominus \text{ve (local maxima)}$$

$$x = 2\sqrt{2} \quad \frac{d^2(D^2)}{dx^2} = 16 = \oplus \text{ve (local minima)}$$

$$\underline{x = 2\sqrt{2}}$$

D^2 minima, (D) minima
Shortest.

Point B $(x, \frac{x^2}{2})$
 $x = 2\sqrt{2}$
 $B(2\sqrt{2}, \frac{8}{2})$
 $(2\sqrt{2}, 4)$

Q-28 For all real values of x , the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is - (A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

Ans. $f(x) = \frac{1-x+x^2}{1+x+x^2}$ $\left(\frac{u}{v}\right)' = \frac{u \cdot v' - u \cdot v''}{v^2}$

$$f'(x) = \frac{(-1+2x) \cdot (1+x+x^2) - (1-x+x^2) \cdot (1+2x)}{(1+x+x^2)^2}$$

$$f'(x) = \frac{-1-x-x^2 + 2x + 2x^2 + 2x^3 - 1 + x - x^2 - 2x + 2x^2 - 2x^3}{(1+x+x^2)^2}$$

$$f'(x) = \frac{2x^2 - 2}{(1+x+x^2)^2} = \frac{2(x^2 - 1)}{(1+x+x^2)^2}$$

$$f'(x) = \frac{2(x+1)(x-1)}{(1+x+x^2)^2} = 0 \text{ (For critical points)}$$

$$\rightarrow x = -1, x = 1$$

First order Derivative Test

0.9
2
1.1

(-1)
(0.9)
(1)

$x=1$

$x=-1$

⊖ ↘ ⊕
minima

⊕ ↗ ⊖
max.

$$f'(-1^-) = \frac{(+).(-)}{(+)} = \ominus$$

Dec.

$$f'(-1^-) = \frac{(-).(-)}{(+)} = \oplus$$

$$f'(-1^+) = \frac{(+).(-)}{(+)} = \ominus$$

⊖ ↘ ⊕ ↗ ⊖
maxima

$$f'(1^+) = \frac{+.+}{+} = \oplus$$

⊖ ↘ ⊕ ↗ ⊖
⊖ ↘ ⊕ ↗ ⊖
minima

∴ minimum value at $x = 1$

$$f(1) = \frac{1-1+1^2}{1+1+1^2} = \frac{1}{3}$$

Q.29) The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \leq x \leq 1$ is — (A) $(\frac{1}{3})^{\frac{1}{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Ans.

maximum, in closed interval

$$x \in [0, 1]$$

Absolute maxima
(Global maxima)

$$f(x) = (x(x-1)+1)^{\frac{1}{3}} = (x^2 - x + 1)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3} (x^2 - x + 1)^{-\frac{2}{3}} \cdot (2x - 1)$$

$$f'(x) = \frac{(2x-1)}{3(x^2-x+1)^{\frac{2}{3}}} = 0 \quad (\text{for critical points})$$

$$x = \frac{1}{2} \in [0, 1]$$

$$f(x) = [x(x-1)+1]^{\frac{1}{3}}$$

$$f(0) = (1)^{\frac{1}{3}} = 1$$

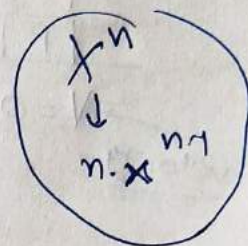
$$f(1) = (1)^{\frac{1}{3}} = 1$$

$$f\left(\frac{1}{2}\right) = \left[\frac{1}{2}\left(\frac{1}{2}-1\right)+1\right]^{\frac{1}{3}} = \left[\frac{1}{2} \times \left(-\frac{1}{2}\right) + 1\right]^{\frac{1}{3}}$$

$$= \left(-\frac{1}{4} + 1\right)^{\frac{1}{3}} = \left(\frac{3}{4}\right)^{\frac{1}{3}}$$

$$\left(\frac{3}{4}\right)^{\frac{1}{3}} < 1 = 1$$

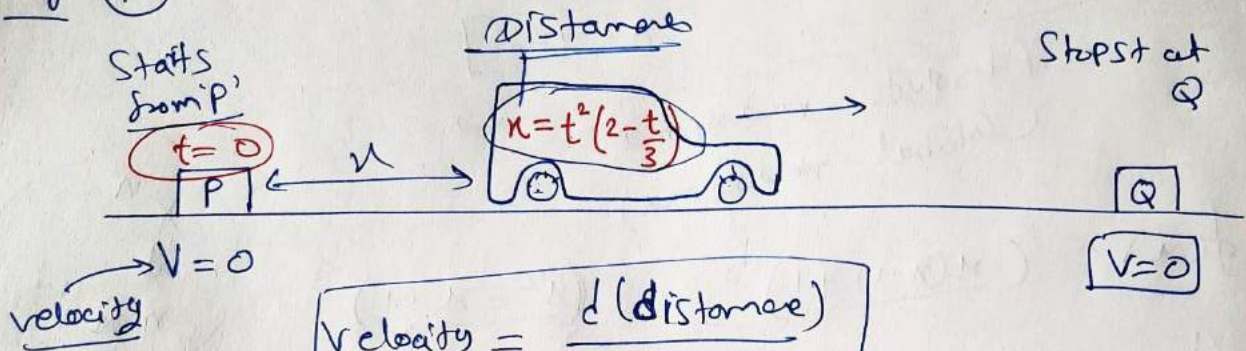
maximum



Miscellaneous Examples Chapter 6

42, 43, 44

e.g. 42



$$\text{Velocity} = \frac{d(\text{Distance})}{d(\text{time})}$$

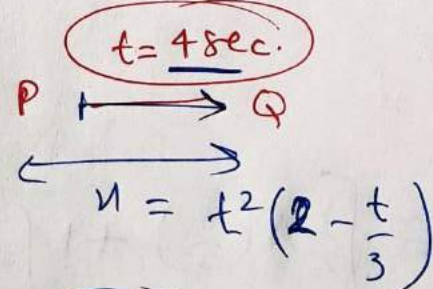
$$V = \frac{d(x)}{dt} = \frac{d(2t^2 - \frac{t^3}{3})}{dt} = 4t - t^2$$

$$V = t(4-t)$$

at P & Q $v=0$

$$\Rightarrow t(4-t) = 0$$

$t=0$, $t=4$
 (P), (Q)



$t = 4$

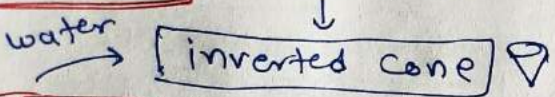
$$x = 4^2(2 - \frac{4}{3})$$

$$PQ = x = 16(\frac{2}{3}) = \underline{\underline{\frac{32}{3} \text{ m}}}$$

e.g. 43

water tank

Semi-vertical angle



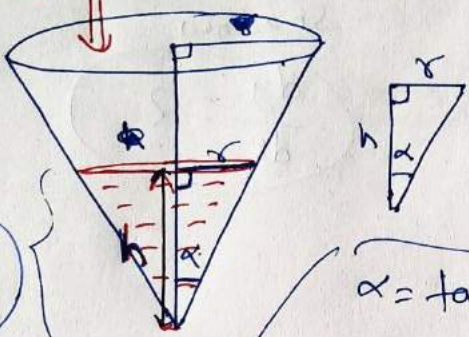
$$\alpha = \tan^{-1}(0.5)$$

5 m³/hour

water level rising rate = ? when depth of water = 4m = h

$\frac{dv}{dt}$
Solution.

$$\frac{dv}{dt} = 5$$



Rate of Change

- D → Diagram
- D → Derivative (v/?)
- F → Formula
- D → Differentiation
- C → Constants

$$\frac{dh}{dt} = ?$$

$$\alpha = \tan^{-1}(0.5)$$

$$\tan \alpha = \frac{1}{2} = \frac{\text{Perp.}}{\text{Base}} = \frac{r}{h}$$

$$\frac{1}{2} = \frac{r}{h}$$

$$h = 2r$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi r^2 (2r)$$

$$V = \frac{2}{3} \pi r^3$$

$$V = \frac{2}{3} \pi \left(\frac{h}{2}\right)^3 = \frac{\pi}{12} h^3$$

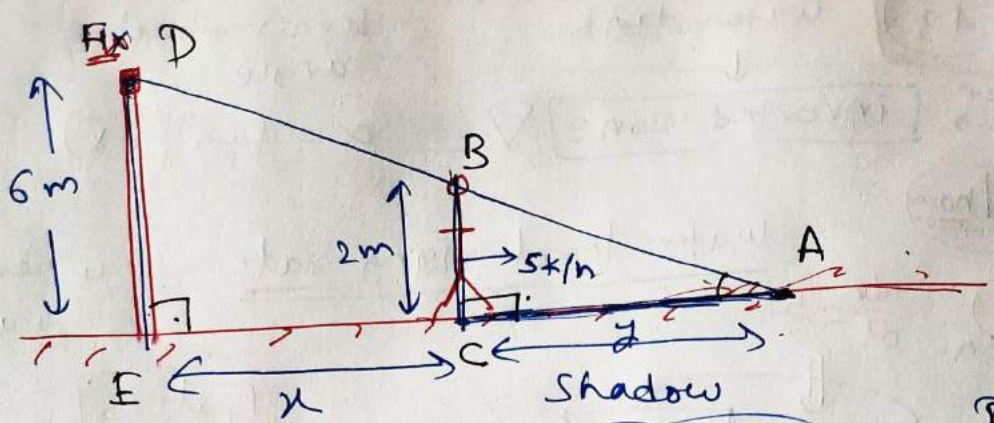
by Diff.

$$\left(\frac{dv}{dt}\right) = \frac{\pi}{12} \times 3h^2 \times \left(\frac{dh}{dt}\right)$$

$$\Rightarrow 5 = \frac{\pi}{12} \times 3(4)^2 \times \left(\frac{dh}{dt}\right)$$

$$\Rightarrow \frac{dh}{dt} = \frac{5 \times 12 \times 7}{22 \times 8 \times 4 \times 4} = \left(\frac{35}{88}\right) \text{ m/h}$$

e.g. (44)



$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = ?$$

- D ✓
- P ✓
- F ✓
- C ✓

(AA)

$$\triangle ACB \sim \triangle AED$$

$$\Rightarrow \frac{AC}{AE} = \frac{CB}{ED}$$

$$\Rightarrow \frac{y}{x+y} = \frac{2}{6}$$

$$\Rightarrow 3y = x+y$$

$$\Rightarrow \boxed{2y = x}$$

$$\boxed{2y = x}$$

$$\Rightarrow 2 \frac{dy}{dt} = \frac{dx}{dt} \rightarrow 5$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{5}{2} \text{ K/h}}$$

Shadow at Rate of Change

Miscellaneous Examples

Chapter 6

46, 48, 51

Example: 46 Find the equation of tangents to the curve $y = \cos(x+y)$, $-2\pi \leq x \leq 2\pi$ that are parallel to the line $x+2y=0$.

Solution:

$$m_T = -\frac{1}{2} = \frac{dy}{dx}$$

Point. \longleftrightarrow

$$x+2y=0$$

$$2y = -x$$

$$y = -\frac{x}{2}$$

$$m = -\frac{1}{2}$$

$$y = m_1x + c$$

$$y = m_2x + c$$

$$m_1 = m_2$$

Curve $y = \cos(x+y)$ diff.

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y) \cdot x \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y) - \sin(x+y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} (1 + \sin(x+y)) = -\sin(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x+y)}{1 + \sin(x+y)} = \text{slope of tangent}$$

$$\left(m_T = -\frac{1}{2} \right)$$

Equate

$$\Rightarrow \frac{-\sin(x+y)}{1 + \sin(x+y)} = -\frac{1}{2}$$

$$\Rightarrow 2\sin(x+y) = 1 + \sin(x+y)$$

$$\Rightarrow \boxed{\sin(x+y) = 1}$$

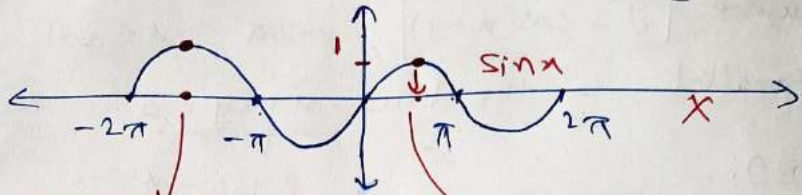
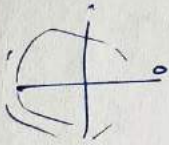
$$\cos(x+y) = \sqrt{1 - \sin^2(x+y)} = \sqrt{1 - 1^2} = 0$$

$$\cos(x+y) = 0$$

$$\text{Cumm } y = \cos(x+y)$$

$$\Rightarrow \boxed{y=0}$$

Point at y -coordinate



$$x = -\frac{3\pi}{2}$$

$$x = \frac{\pi}{2}$$

Points

$$\left(-\frac{3\pi}{2}, 0\right)$$

$$\left(\frac{\pi}{2}, 0\right)$$

$$m_T = -\frac{1}{2}$$

$$m_T = -\frac{1}{2}$$

$$(y - y_0) = m_T(x - x_0)$$

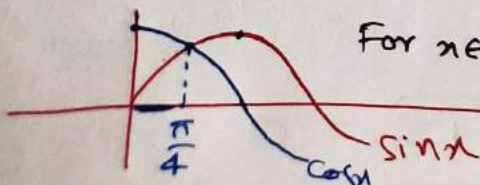
$$\boxed{(y - 0) = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)}$$

$$\boxed{(y - 0) = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)}$$

Example 48 Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$, is always an increasing function in $(0, \frac{\pi}{4})$.

Ans. $f(x) = \tan^{-1}(\sin x + \cos x)$ $x \in (0, \frac{\pi}{4})$ $f'(x) > 0$
+
+

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x) = \frac{+}{+} = +ve.$$



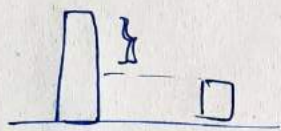
For $x \in (0, \frac{\pi}{4})$ $\boxed{\cos x > \sin x}$

$$\frac{f'(x) = +ve}{\text{Increasing}}$$

Example (5) Manufacturer can sell x items at a price of rupees $(5 - \frac{x}{100})$ each. The cost price of x items in Rs $(\frac{x}{5} + 500)$. Find the number of items he should sell to earn maximum profit.

Ans.

Profit = S.P. - ~~CP~~ CP.



S.P. = $S(x)$

= $(5 - \frac{x}{100})x$
for 1 item x -items

= $(5x - \frac{x^2}{100})$

CP. = $C(x) = (\frac{x}{5} + 500)$

Profit = $P(x) = S(x) - C(x)$

$P(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$

$P(x) = \frac{24x}{5} - \frac{x^2}{100} - 500$

maxima

$P'(x) = \frac{24}{5} - \frac{x}{50} = 0$ (For Critical Points)

$\Rightarrow \frac{24}{5} = \frac{x}{50}$

$\Rightarrow x = 240$

SODT

$P''(x) = 0 - \frac{1}{50}$

$P''(240) = -\frac{1}{50} = \ominus ve$

(local maxima)

\therefore Manufacturer should sell 240 items to earn maximum profit.

Miscellaneous Exercise on Chapter 6

Q.1 Using differentials, find the approximate value of the following:

(a) $(\frac{17}{81})^{\frac{1}{4}}$

(b) $(33)^{\frac{1}{5}}$

Ans.

(a) $(\frac{17}{81})^{\frac{1}{4}}$

Function: $y = f(x) = x^{\frac{1}{4}}$

$81 = 3^4$

$16 = 2^4$

$y = (\frac{16}{81})^{\frac{1}{4}} = \frac{2}{3}$

$(\frac{17}{81})^{\frac{1}{4}} = (\frac{16}{81} + \frac{1}{81})^{\frac{1}{4}} = \frac{2}{3} + \Delta y$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Differentiate

$\therefore y = x^{\frac{1}{4}}$

$\frac{dy}{dx} = \frac{1}{4} x^{\frac{1}{4}-1}$

$= \frac{1}{4} x^{-\frac{3}{4}}$

$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$

$(x = \frac{16}{81})$

$\frac{dy}{dx} = \frac{1}{4(\frac{16}{81})^{\frac{3}{4}}}$

$= \frac{1}{4(\frac{2}{3})^3}$

$\Delta y = \frac{dy}{dx} \times \Delta x$

$\Delta y = (\frac{-21}{4 \times 8}) \times \frac{1}{81} = \frac{1}{32 \times 3}$

$\Delta y = \frac{1}{96} = 0.0103$

$(\frac{17}{81})^{\frac{1}{4}} = \frac{2}{3} + \Delta y$

$= 0.6666... + 0.0103$

$= 0.667 + 0.010$

$= 0.677$

(b) $(33)^{\frac{1}{5}}$

$y = f(x) = x^{-\frac{1}{5}}$

$2^5 = 32$
 $2 = (32)^{\frac{1}{5}}$

$x = 32, \Delta x = 1$

$y = x^{-\frac{1}{5}} \quad (x = 32)$

$y = (32)^{-\frac{1}{5}}$

$y = (2^5)^{-\frac{1}{5}} = 2^{-1} = \frac{1}{2} = 0.5$

$(x + \Delta x)^{-\frac{1}{5}} = y + \Delta y$
 $(32 + 1)^{-\frac{1}{5}} = ? = 0.5 + \Delta y$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$\Rightarrow \Delta y = \frac{dy}{dx} \cdot \Delta x$

$y = x^{-\frac{1}{5}}$

$\frac{dy}{dx} = -\frac{1}{5} \cdot x^{-\frac{1}{5}-1} = -\frac{1}{5} \cdot x^{-\frac{6}{5}}$

$= -\frac{1}{5 \cdot x^{\frac{6}{5}}}$

$\frac{dy}{dx} = -\frac{1}{5 \cdot (32)^{\frac{6}{5}}}$

$= -\frac{1}{5 \cdot (2^5)^{\frac{6}{5}}}$

$= -\frac{1}{5 \cdot 64} = -\frac{1}{320}$

$\Delta y = \left(-\frac{1}{320}\right) \cdot (1)$
 $= -\frac{1}{320}$

~~...~~
 ≈ -0.003

$(33)^{\frac{1}{5}}$

$= y + \Delta y$

$= 0.5 - 0.003$

$= 0.500$

$- 0.003$

$= 0.497$

$\frac{1}{2} = 0.5$

$\frac{1}{4} = 0.25$

$\frac{1}{8} = 0.125$

$\frac{1}{16} = 0.0625$

$\frac{1}{32} = 0.03125$

$\frac{1}{320} = 0.003125$

$\rightarrow = 0.003$

Miscellaneous Exercise on Chapter 6

Q.2 Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x=e$.

Solution,

$$f(x) = \frac{\log x}{x}$$

$$\left(\frac{y}{v}\right)'$$

Domain = $\log x$
 $x > 0$

$$f'(x) = \frac{\left(\frac{1}{x}\right) \cdot x - (\log x) \cdot 1}{x^2}$$

$$f'(x) = \frac{1 - \log x}{x^2} = 0 \quad (\text{For Critical Points.})$$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow 1 = \log x$$

$$\Rightarrow 1 = \log_e x$$

EXP

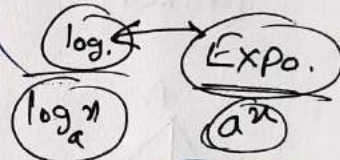
$$\Rightarrow (e)^{(1)} = x$$

$$\Rightarrow \boxed{x=e}$$

Critical point

maths

$$\log a = \log_e a$$



$$\log_a(b) = 100$$

$$b = (a)^{100}$$

SODT

$$f'(x) = \frac{(1 - \log x)}{x^2}$$

put $x=e$

$$f''(x) = \frac{\left(-\frac{1}{x}\right) \cdot x^2 - (1 - \log x) \cdot 2x}{(x^2)^2}$$

$$\log_e e = 1$$

$x=e$
Point of maxima

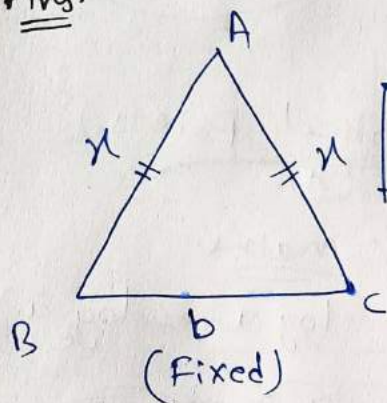
$$f''(e) = \frac{-e - 0}{e^4} = -\frac{1}{e^3} = -ve$$

local maxima

Miscellaneous Exercise on chapter 6

Q.3 Two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base?

Ans.



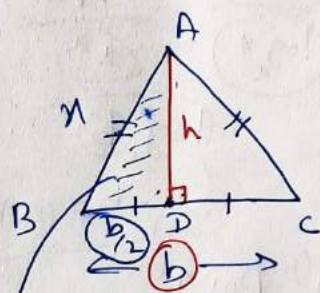
$$\frac{dx}{dt} = -3$$

$$\frac{dA}{dt} = ?$$

when $x = y = b$

Rate of change

- ✓ D → Diagram
- ✓ D → Derivative
- ✓ F → Formula
- ✓ D → Differentiation
- ✓ C → Constant



P.G.T

$$x^2 = \left(\frac{b}{2}\right)^2 + h^2$$

$$\Rightarrow h = \sqrt{x^2 - \frac{b^2}{4}}$$

$$\text{Area} = \frac{1}{2} (\text{Base}) \times (\text{height})$$

$$A = \frac{1}{2} (b) \times (h)$$

$$A = \frac{1}{2} b \sqrt{x^2 - \frac{b^2}{4}}$$

by diff. w.r. to 't'

$$\frac{dA}{dt} = \frac{b}{2} \cdot \frac{x \cdot \frac{dx}{dt}}{\sqrt{x^2 - \frac{b^2}{4}}} \cdot \frac{dx}{dt}$$

when $x = b$

$$\frac{dA}{dt} = \frac{b}{2} \times \frac{b}{\sqrt{b^2 - \frac{b^2}{4}}} \cdot (-3)$$

$$\frac{dA}{dt} = -\sqrt{3} b$$

Area decreases

at $\sqrt{3} b \text{ cm}^2/\text{s}$

Miscellaneous Exercise on Chapter 6

Q.4 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point $(1, 2)$.

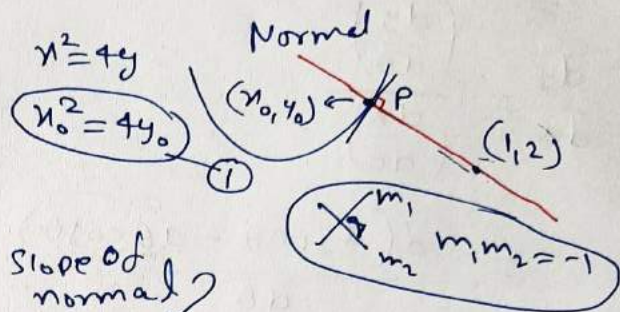
Ans.

$$x^2 = 4y$$

$$\Rightarrow 2x = 4 \frac{dy}{dx}$$

$$\Rightarrow \left[\frac{dy}{dx} = \frac{x}{2} \right] \text{ Slope of tangent}$$

$$\text{Slope of tangent at } (x_0, y_0) = \frac{x_0}{2} = m_T$$



$$\text{Slope of normal } m_N \cdot m_T = -1$$

$$m_N = -\frac{1}{m_T} = -\frac{2}{x_0}$$

Equation of normal $(y - y_0) = m_N (x - x_0)$

$$\Rightarrow y - y_0 = -\frac{2}{x_0} (x - x_0)$$

which passes through $(1, 2)$

put $x=1$
 $y=2$

$$\Rightarrow 2 - y_0 = -\frac{2}{x_0} (1 - x_0)$$

$$\Rightarrow 2x_0 - y_0 x_0 = -2 + 2x_0$$

$$\Rightarrow x_0 y_0 = 2 \quad \text{--- (2)} \rightarrow y_0 = \frac{2}{x_0}$$

$$x_0^2 = 4y_0 \quad \text{--- (1)}$$

by eqn (1) & (2) -

$$\Rightarrow x_0^2 = 4 \left(\frac{2}{x_0} \right)$$

$$\Rightarrow x_0^3 = 8 = 2^3$$

$$x_0 = 2$$

$$x_0^2 = 4y_0$$

$$\Rightarrow 2^2 = 4y_0$$

$$y_0 = 1 \quad \text{Point } (2, 1)$$

$$\text{Normal: } (y - 1) = -\frac{2}{2} (x - 2)$$

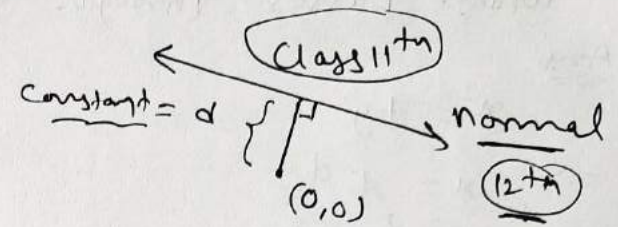
$$\Rightarrow y + x - 3 = 0$$

Q.5 Show that the normal at any point ' θ ' to the curve $x = a \cos \theta + a \theta \sin \theta$, $y = a \sin \theta - a \theta \cos \theta$ is at a constant distance from origin.

Ans.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{d(a \sin \theta - a \theta \cos \theta)}{d\theta} \bigg/ \frac{d(a \cos \theta + a \theta \sin \theta)}{d\theta}$$



$$= \frac{a \cos \theta - a \cos \theta + a \theta \sin \theta}{-a \sin \theta + a \sin \theta + a \theta \cos \theta}$$

$$\frac{dy}{dx} = \frac{\sin \theta}{\cos \theta}$$

Slope of tangent (m_T)

Slope of normal $= m_N = -\frac{1}{m_T}$

$$m_N = -\frac{\cos \theta}{\sin \theta}$$

Point $(\underbrace{a \cos \theta + a \theta \sin \theta}_{x_0}, \underbrace{a \sin \theta - a \theta \cos \theta}_{y_0})$

Equation of normal.

$$(y - a \sin \theta + a \theta \cos \theta) = \frac{-\cos \theta}{\sin \theta} (x - a \cos \theta - a \theta \sin \theta)$$

$$\Rightarrow y \sin \theta - a \sin^2 \theta + a \theta \cos \theta \sin \theta = -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta$$

$$\Rightarrow x \cos \theta + y \sin \theta - a (\sin^2 \theta + \cos^2 \theta) = 0$$

$$\Rightarrow \boxed{x \cos \theta + y \sin \theta - a = 0} \quad \text{Normal,}$$

Normal

$$x \cos \theta + y \sin \theta - a = 0$$

origin (0,0)

$$d = \left| \frac{0 + 0 - a}{\sqrt{\cos^2 \theta + \sin^2 \theta}} \right| = \left| \frac{-a}{\sqrt{1}} \right|$$

$$d = |-a| = a$$

↑ Constant

Distance b/w
normal and origin

Distance Formula

b/w line & point

$ax + by + c = 0$ (x_0, y_0)

$$d = \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$$

Q.6 Find the intervals in which the function

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{(2 + \cos x)}$$

is (i) increasing. $f'(x) \oplus$
(ii) decreasing. $f'(x) \ominus$

Ans.

$$f(x) = \frac{4 \sin x - x(2 + \cos x)}{(2 + \cos x)}$$

$$f(x) = \frac{4 \sin x}{2 + \cos x} - x$$

$$f'(x) = \frac{(4 \cos x)(2 + \cos x) - 4 \sin x (-\sin x)}{(2 + \cos x)^2} \quad \ominus \quad -$$

$$f'(x) = \frac{8 \cos x + \cancel{4 \cos^2 x} + \cancel{4 \sin^2 x} - 4 + \cos^2 x - 4 \cos x}{(2 + \cos x)^2}$$

$$f'(x) = \frac{4 \cos x - \cos^2 x}{(2 + \cos x)^2} = \frac{\cos x (4 - \cos x)}{(2 + \cos x)^2}$$

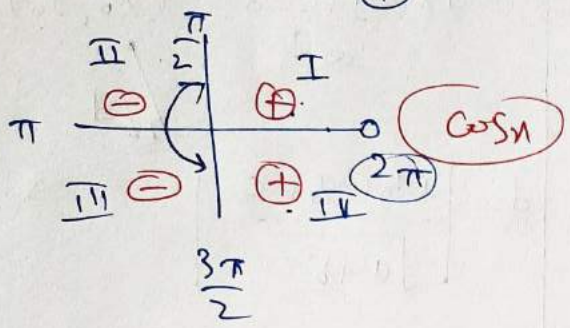
$$f'(x) = \frac{\cos x (4 - \cos x)}{(2 + \cos x)^2}$$

Factors

$$(2 + \cos x)^2 = \text{Always } \oplus$$

$$(4 - \cos x) = \text{Always } \oplus$$

$$\cos x \rightarrow \oplus \text{ or } \ominus$$



$x \in (\frac{\pi}{2}, \frac{3\pi}{2}) \rightarrow \cos x = \ominus \text{ve} \rightarrow f'(x) = \ominus \text{ve}$
(Decreasing)

$x \in (0, \frac{\pi}{2})$ or $x \in (\frac{3\pi}{2}, 2\pi) \rightarrow \cos x = \oplus \text{ve} \rightarrow f'(x) = \oplus \text{ve}$
Increasing

Increasing

$$x \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$$

Decreasing

$$x \in (\frac{\pi}{2}, \frac{3\pi}{2})$$

Q.7 Find the interval in which the function $f(x) = x^3 + \frac{1}{x^3}$, ($x \neq 0$) is (I) increasing, $f'(x) = \oplus$
(II) Decreasing, $f'(x) = \ominus$

Ans.

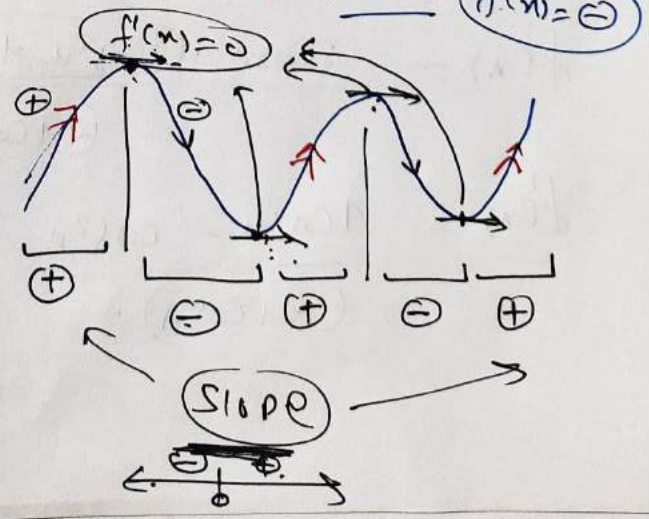
$$f'(x) = 3x^2 - \frac{3}{x^4}$$

$$f'(x) = 3 \left(\frac{x^6 - 1}{x^4} \right) = 0 \text{ (C.P.)}$$

$$x^6 - 1 = 0$$

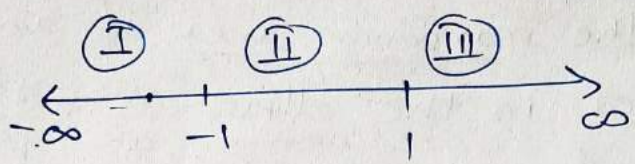
$$x^6 = 1$$

$$x = \pm 1$$



$$f'(x) = \frac{3(x^6 - 1)}{x^4}$$

Critical points
 $x = \pm 1$ where $f'(x) = 0$



Interval	$f'(x)$ sign	\uparrow / \downarrow
I $(-\infty, -1)$	\oplus	\uparrow (Increasing)
II $(-1, 1)$	\ominus	\downarrow (Dec.)
III $(1, \infty)$	\oplus	\uparrow (Increasing)

Increasing

$$x \in (-\infty, -1) \cup (1, \infty)$$

$$\underline{x < -1} \text{ or } \underline{x > 1}$$

Decreasing

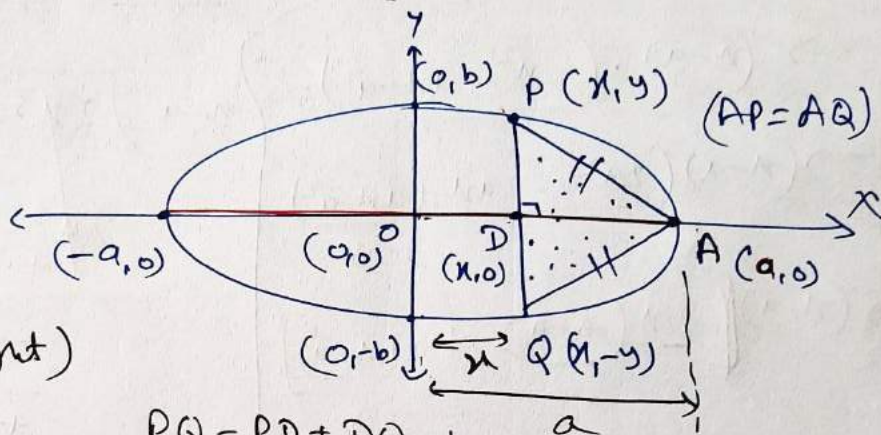
$$x \in (-1, 1)$$

$$\underline{-1 < x < 1}$$

Miscellaneous Exercise on Chapter 6

Q.8 Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Ans.



Area(ΔAPQ)

$= A = \frac{1}{2} (\text{Base})(\text{Height})$

$A = \frac{1}{2} (PQ)(DA)$

$PQ = PD + DQ$

$= |y| + |y|$
 $= 2y$

$AD = (a - x)$

$A = \frac{1}{2} (2y)(a - x)$

$(A) \uparrow \quad (A^2) \uparrow$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2 \left(1 - \frac{x^2}{a^2} \right)$

$A^2 = y^2 (a - x)^2 = b^2 \left(\frac{a^2 - x^2}{a^2} \right) \cdot (a^2 + x^2 - 2ax)$

$A^2 = \frac{b^2}{a^2} (a^4 + a^2 x^2 - 2a^3 x - a^2 x^2 - x^4 + 2ax^3)$

$\frac{d(A^2)}{dx} = \frac{b^2}{a^2} (-2a^3 - 4x^3 + 6ax^2)$

$\frac{d(A^2)}{dx} = -2 \frac{b^2}{a^2} (2x^3 - 3ax^2 + a^3) = 0$ (for Critical Points)

$$\frac{d(A^2)}{dx} = -\frac{2b^2}{a^2} (2x^3 - 3ax^2 + a^3) = 0$$

$$2x^3 - 3ax^2 + a^3 = 0$$

$x=a$ satisfy

$(x-a)$ factor

$$\Rightarrow (x-a) \cdot (2x^2 - ax - a^2) = 0$$

$$\Rightarrow (x-a) (2x^2 - 2ax + ax - a^2) = 0$$

$$\Rightarrow (x-a) [2x(x-a) + a(x-a)] = 0$$

$$\Rightarrow (x-a)^2 (2x+a) = 0$$

$x=a$

$-\frac{a}{2}$

Critical Points

SODT

$$\frac{d(A^2)}{dx} = -\frac{2b^2}{a^2} (2x^3 - 3ax^2 + a^3)$$

$$\frac{d^2(A^2)}{dx^2} = -\frac{2b^2}{a^2} (6x^2 - 6ax)$$

Put $(x = -\frac{a}{2})$

$$\frac{d^2(A^2)}{dx^2} = \ominus (\oplus + \oplus)$$

$= \ominus ve$

Maxima

$$\begin{array}{r} 2x^3 - 3ax^2 + a^3 \\ \underline{- 2x^3 - 2ax^2} \\ - ax^2 + a^3 \\ \underline{+ ax^2 - a^2x} \\ - a^2x + a^3 \\ \underline{+ a^2x - a^3} \\ 0 \end{array}$$

$x = -\frac{a}{2}$ Maxima

$$A^2 = \frac{b^2}{a^2} (a^4 - 2a^3x - x^4 + 2ax^3)$$

$$A^2 = \frac{b^2}{a^2} (a^4 + a^4 - \frac{a^4}{16} - \frac{a^4}{4})$$

$$A^2 = \frac{b^2}{a^2} \left(\frac{32 - 1 - 4}{16} \right) a^4$$

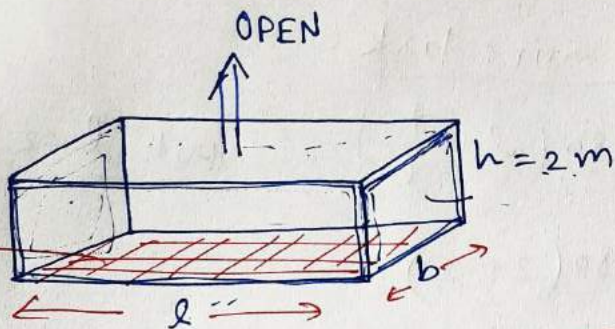
$$A^2 = \frac{27}{16} a^2 b^2 \uparrow$$

$$A = \frac{3\sqrt{3}}{4} ab \uparrow$$

Q.9

Cuboid

l x b



$$\text{Volume} = 8 \text{ m}^3$$

$$lbh = 8$$

$$lb(2) = 8$$

$$lb = 4 = \text{Area (Base)}$$

$$\text{Total cost} = \text{Area} \times \text{Rate}$$

$$\text{Rate of cost for base} = ₹ 70/\text{m}^2 \rightarrow \text{Total cost of base} = 4 \times 70$$

$$\text{Rate of cost for sides (walls)} = ₹ 45/\text{m}^2 \rightarrow \text{Total cost of sides} = 4(l+b) \cdot 45$$

$$\begin{aligned} \text{Area of sides} &= 2(lh) + 2(bh) = 2h(l+b) \\ &= 4(l+b) \end{aligned}$$

least

$$\text{Total Expenditure on tank} = E = 280$$

$$E = 280 + 180(l+b)$$

$$+ 180(l+b)$$

$$E = 280 + 180\left(l + \frac{4}{l}\right)$$

$$\begin{aligned} \because lb &= 4 \\ b &= \frac{4}{l} \end{aligned}$$

$$\frac{d(E)}{dl} = 180\left(1 - \frac{4}{l^2}\right) = 0 \quad (\text{for critical points})$$

S.O.P.T

$$1 = \frac{4}{l^2}$$

$$l^2 = 4 \Rightarrow l = \pm 2$$

$$\begin{aligned} l &= -2 \times \\ l &= 2 \end{aligned}$$

$$\frac{d^2(E)}{dl^2} = 180\left(\frac{8}{l^3}\right)$$

$$\text{If } l = 2, \frac{d^2(E)}{dl^2} = (+)$$

$l = 2$ Point of minima

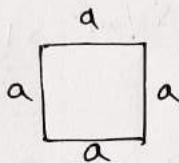
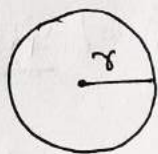
Cost of least expensive tank

$$E = 280 + 180 \left(2 + \frac{4}{l} \right) \quad (\text{put } \underline{l=2})$$

$$E = 280 + 180(2+2)$$

$$E = 280 + 720 = \underline{\underline{1000 \text{ ₹}}}$$

Q.10



To Prove
 $a = 2r$

$$2\pi r + 4a = K \quad (\text{Constant})$$

$$A = \pi r^2 + a^2$$

$$a = \frac{K - 2\pi r}{4}$$

$$A = \pi r^2 + \left(\frac{K - 2\pi r}{4} \right)^2 = \pi r^2 + \frac{K^2 + 4\pi^2 r^2 - 4\pi K r}{16}$$

$$A = \frac{16\pi r^2 + K^2 + 4\pi^2 r^2 - 4\pi K r}{16}$$

$$\frac{dA}{dr} = \frac{32\pi r + 8\pi^2 r - 4\pi K}{16} = 0 \quad (\text{For C.P.})$$

S.O.D.T.

$$4\pi(8r + 2\pi r - K) = 0$$

$$\Rightarrow \underline{\underline{r = \frac{K}{8 + 2\pi}}}$$

$$\frac{d^2A}{dr^2} = \frac{32\pi + 8\pi^2}{16}$$

= (+) minima

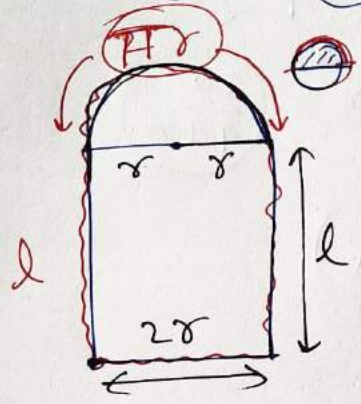
we have

$$2\pi r + 4a = K$$

$$r = \frac{K}{8+2\pi}$$

$$r = \frac{2\pi r + 4a}{8+2\pi} \Rightarrow 8r + 2\pi r = 2\pi r + 4a$$
$$\Rightarrow 2r = a$$

Q.11



total perimeter = 10 m.

$$2l + 2r + \pi r = 10$$

$$l = \frac{10 - 2r - \pi r}{2}$$

Maximum light passing = maximum area

A = Area of rectangle + Area of circle

$$A = 2rl + \frac{\pi r^2}{2}$$

$$A = 2r \left(\frac{10 - 2r - \pi r}{2} \right) + \frac{\pi r^2}{2}$$

$$A = 10r - 2r^2 - \pi r^2 + \frac{\pi r^2}{2}$$

$$A = 10r - 2r^2 - \frac{\pi r^2}{2}$$

$$\frac{dA}{dr} = 10 - 4r - \pi r = 0$$

For C.P.

$$10 = r(4 + \pi)$$

$$r = \frac{10}{4 + \pi}$$

$$\frac{dA}{dr} = 10 - 4r - \pi r$$

C.P.

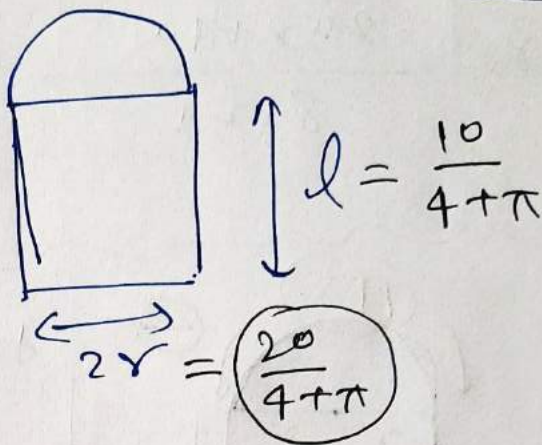
$$r = \frac{10}{4 + \pi}$$

$$\frac{d^2A}{dr^2} = -4 - \pi$$

Point of maxima

= \ominus ve

maxima



$$l = \frac{10 - 2r - \pi r}{2} \quad \left(\text{Put } r = \frac{10}{4 + \pi} \right)$$

$$l = \frac{10 - \frac{20}{4 + \pi} - \frac{10\pi}{4 + \pi}}{2} = \frac{40 + 10\pi - 20 - 10\pi}{2(4 + \pi)}$$

$$l = \frac{10}{4 + \pi}$$

Miscellaneous Exercise on Chapter - 6

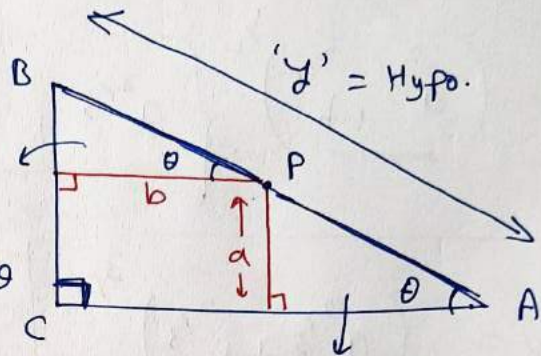
Q.12 A point on the hypotenuse of a triangle is at a distance a and b from the sides of the triangle. Show that the minimum length of hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.

Ans.

$y = AP + PB$

$y = a \operatorname{cosec} \theta + b \sec \theta$

$\cos \theta = \frac{b}{BP}$
 $BP = b \sec \theta$



$\sin \theta = \frac{a}{AP}$

$AP = a \operatorname{cosec} \theta$

$\frac{dy}{d\theta} = -a \operatorname{cosec} \theta \cdot \cot \theta + b \sec \theta \cdot \tan \theta = 0$ (For Critical Points)

SODT

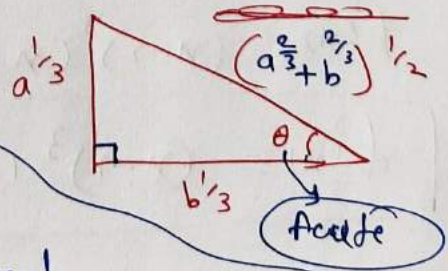
$b \cdot \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = a \cdot \frac{1}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta}$

$\Rightarrow b \cdot \frac{\sin^3 \theta}{\cos^3 \theta} = a \frac{\cos^3 \theta}{\sin^3 \theta}$

$\Rightarrow \tan^3 \theta = \frac{a}{b}$

$\tan \theta = \frac{a^{1/3}}{b^{1/3}} = \frac{\text{Perp.}}{\text{Base}}$

$\frac{d^2y}{d\theta^2} = +a \operatorname{cosec} \theta \cdot \cot^2 \theta + a \operatorname{cosec} 3\theta + b \sec \theta \cdot \tan^2 \theta + b \sec^3 \theta$



If we put the values of critical Point we get

$\frac{d^2y}{d\theta^2} = +ve$ (Minimum)

Hypotenuse $y = a \operatorname{cosec} \theta + b \sec \theta$

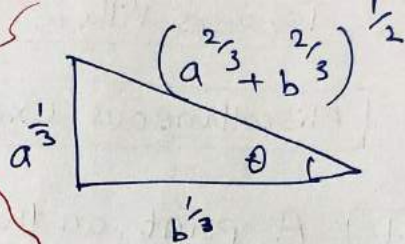
minimum

$$y = a \frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b \frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}}$$

$$y = \left(a^{2/3} + b^{2/3} \right)^{1/2} \cdot \left\{ a^{2/3} + b^{2/3} \right\}^1$$

$$y = \left(a^{2/3} + b^{2/3} \right)^{3/2}$$

Hyp. (minimum).



$$\operatorname{cosec} \theta = \frac{H}{P}$$

$$\sec \theta = \frac{H}{B}$$

Q.13 Find the points at which the function $f(x) = (x-2)^4 \cdot (x+1)^3$ has (i) local minima (ii) local maxima (iii) point of inflection.

~~FODT~~

~~FODT~~

~~SOPT~~

Ans. $f(x) = (x-2)^4 \cdot (x+1)^3$

$$f'(x) = 4(x-2)^3 (x+1)^3 + 3(x-2)^4 (x+1)^2$$

$$f'(x) = (x-2)^3 (x+1)^2 \cdot [4x+4 + 3x-6]$$

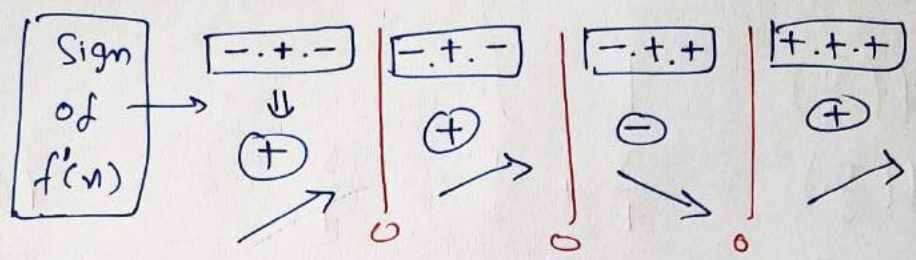
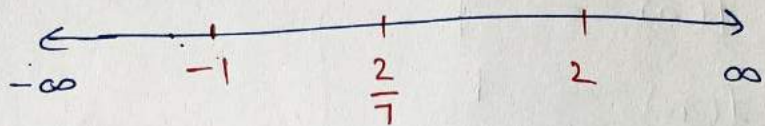
$$f'(x) = (x-2)^3 (x+1)^2 (7x-2) = 0 \quad \left(\text{For Critical Points} \right)$$

$$x=2, \quad x=-1, \quad x=\frac{2}{7}$$

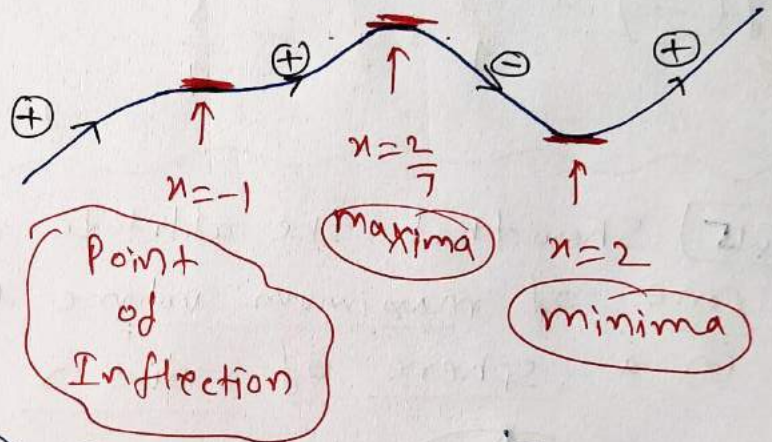
$$f'(x) = (x-2)^3 (x+1)^2 (7x-2) = 0 \rightarrow \boxed{-1, \frac{2}{7}, 2} \rightarrow \underline{f'(x) = 0}$$

Slope

FODT



Graph



Q.14 Find the absolute maximum and minimum values of $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$.

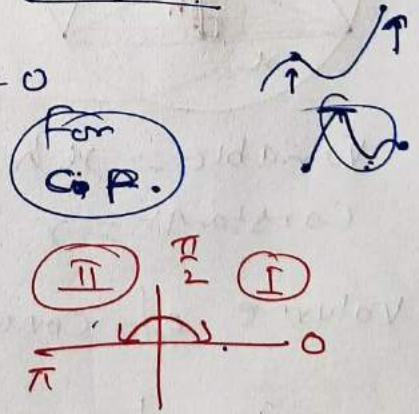
Ans. $f'(x) = 2 \cos x \cdot (-\sin x) + \cos x = 0$

$\cos x (-2 \sin x + 1) = 0$

$\cos x = 0$, $\sin x = \frac{1}{2}$

\downarrow , \downarrow

$x = \frac{\pi}{2}$, $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$



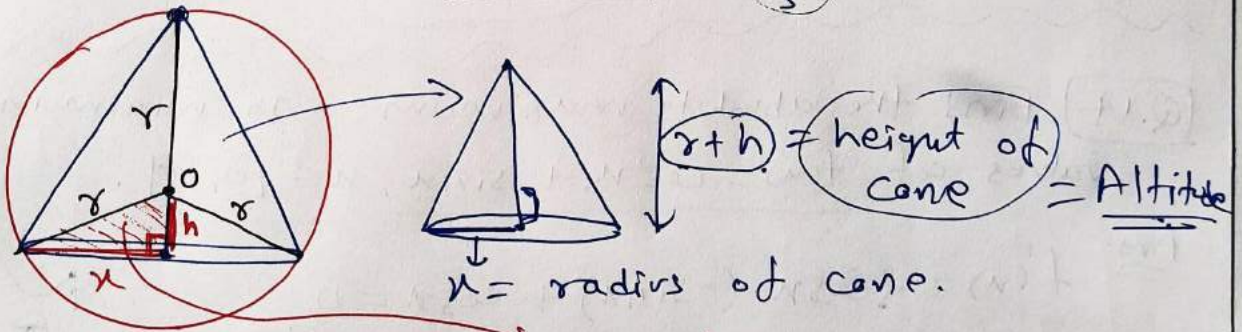
$$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi]$$

$$\left. \begin{aligned} f(0) &= 1 \\ f(\pi) &= 1 \end{aligned} \right\} \rightarrow \text{Absolute minimum} = 1$$

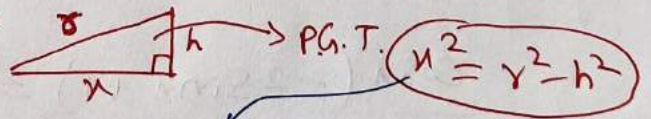
$$\left. \begin{aligned} f\left(\frac{\pi}{2}\right) &= 1 \\ f\left(\frac{\pi}{6}\right) &= \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \\ f\left(\frac{5\pi}{6}\right) &= \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \end{aligned} \right\} \rightarrow \text{Absolute maximum} = \frac{5}{4}$$

Q.15 Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$

Ans.



Variable = x, h
Constant = r



$$\text{Volume of cone} = V = \frac{1}{3} \pi x^2 \cdot (r+h)$$

$$V = \frac{1}{3} \pi (r^2 - h^2) (r+h)$$

$$V = \frac{1}{3} \pi (r^3 + r^2h - rh^2 - h^3)$$

Maxima \rightarrow Diff.

$$V = \frac{1}{3} \pi (r^2 + r^2 h - r h^2 - h^3)$$

↓
variable \rightarrow $(h), V$
Constant $\rightarrow r$

$$\frac{dV}{dh} = \frac{1}{3} \pi (r^2 - 2rh - 3h^2) = 0 \quad \text{For C.P.}$$

$$= -\frac{1}{3} \pi (3h^2 + 2rh - r^2) = 0$$

$$= -\frac{1}{3} \pi (3h^2 + 3rh - rh - r^2) = 0$$

$$\Rightarrow 3h(h+r) - r(h+r) = 0$$

$$(h+r) \cdot (3h-r) = 0$$

$$h = -r$$

$$h = \frac{r}{3} \quad \text{C.P.}$$

$$\frac{d^2V}{dh^2} = \frac{1}{3} \pi (-2r - 6h)$$

Put $h = \frac{r}{3}$ \rightarrow

$$\frac{d^2V}{dh^2} = \frac{1}{3} \pi (-2r - 6 \cdot \frac{r}{3})$$

$$= -ve$$

maxima

$h = \frac{r}{3}$
Point of maxima

Altitude of cone = $r+h$
(Height)

$$= r + \frac{r}{3}$$

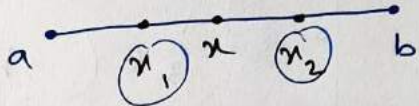
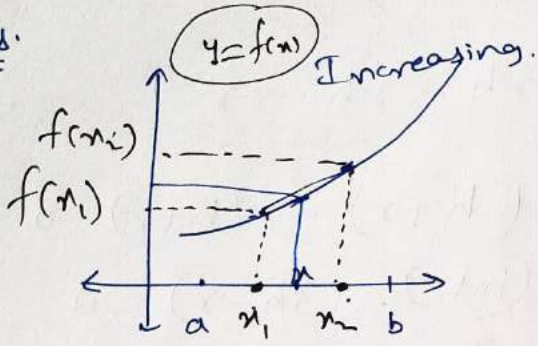
$$= \frac{4r}{3}$$

$$\rightarrow \frac{4r}{3}$$

Miscellaneous Exercise on Chapter 6

Q.16 Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b) .

Ans.



let $x_1, x_2 \in [a, b]$ $x_1 < x_2$

let $x \in [x_1, x_2]$

Mean Value Theorem

Proof: $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\therefore f'(x) > 0$
 $\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$

$\Rightarrow \boxed{f(x_2) - f(x_1) > 0}$

$\boxed{f(x_2) > f(x_1)}$

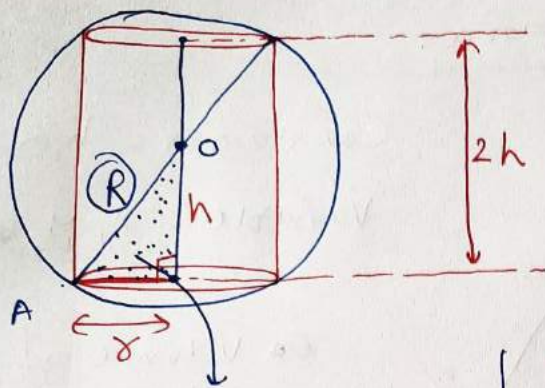
\therefore Here if $x_1 < x_2$ then $f(x_1) < f(x_2)$ } }

\therefore f is increasing function.

To prove: If $x_1 < x_2$
 then $f(x_1) < f(x_2)$
Inc.

Q.17 Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find its maximum volume.

Ans.



P.G.T.

$$R^2 = r^2 + h^2$$

$$\Rightarrow r^2 = R^2 - h^2$$

To Prove:

$$2h = \frac{2R}{\sqrt{3}}$$

Volume of Cylinder

$$V = \pi r^2 (2h)$$

$$V = \pi (R^2 - h^2) \cdot 2h$$

$$V = 2\pi (R^2 h - h^3)$$

maxima

Diff.

For C.P.

$$\frac{dV}{dh} = 2\pi (R^2 - 3h^2) = 0$$

SODT

$$\frac{d^2V}{dh^2} = 2\pi (-6h) = -ve$$

$$h = \frac{R}{\sqrt{3}}$$

maxima

$$R^2 = 3h^2$$

$$\sqrt{\frac{R^2}{3}} = h$$

$$h = \frac{R}{\sqrt{3}}$$

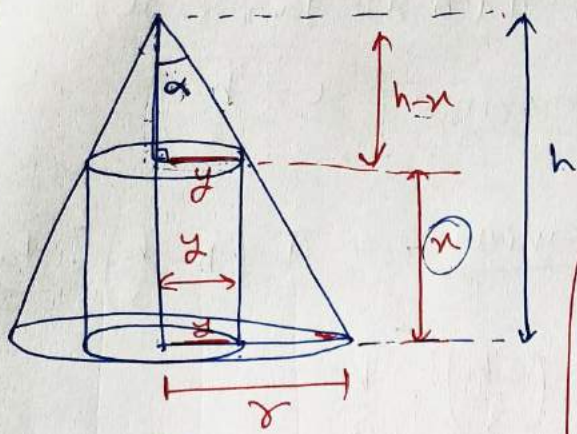
$$\text{Height of cylinder} = 2h = \frac{2R}{\sqrt{3}}$$

$$\text{Volume of cylinder} = V = \pi (R^2 - h^2) 2h$$

$$V = \pi \left(R^2 - \frac{R^2}{3} \right) 2 \times \frac{R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

Q.18

(P/B)



$$\tan \alpha = \frac{y}{h-x} = \frac{y}{h}$$

Small Δ

$$\Rightarrow \begin{cases} y = (h-x) \cdot \tan \alpha \\ r = h \cdot \tan \alpha \end{cases}$$

To prove

$$\text{Height of cylinder} = \frac{1}{3} (\text{Height of cone})$$

$$x = \frac{h}{3}$$

$$V = \frac{4}{27} \pi h^3 \tan^2 \alpha$$

maximum Volume of Cylinder

$$V = \pi y^2 \cdot x$$

$$V = \pi (h-x)^2 \tan^2 \alpha \cdot x$$

$$V = \pi \tan^2 \alpha (h^2 x - 2hx^2 + x^3)$$

Diff.

$$\frac{dV}{dx} = \pi \tan^2 \alpha (h^2 - 4hx + 3x^2)$$

(For Critical Points)

$$\frac{dV}{dx} = 0$$

$$\Rightarrow h^2 - 4hx + 3x^2 = 0$$

$$\Rightarrow h^2 - 3hx - hx + 3x^2 = 0$$

$$\Rightarrow h(h-3x) - x(h-3x) = 0$$

$$\Rightarrow (h-x)(h-3x) = 0$$

$$h = x$$

$$x = \frac{h}{3} \text{ (C.P.)}$$

SODT

↓

$$\frac{d^2V}{dx^2} = \pi \tan^2 \alpha \cdot (-4h + 6x)$$

Put $x = \frac{h}{3}$ → Proved

$$\frac{d^2V}{dx^2} = \pi \tan^2 \alpha \cdot (-4h + 2h)$$

∴ve

maxima

volume of cylinder $V = \pi (h-x)^2 \tan^2 \alpha \cdot x$

point of maxima $x = \frac{h}{3}$

maximum volume of cylinder = $\pi \left(h - \frac{h}{3}\right)^2 \cdot \tan^2 \alpha \cdot \frac{h}{3}$

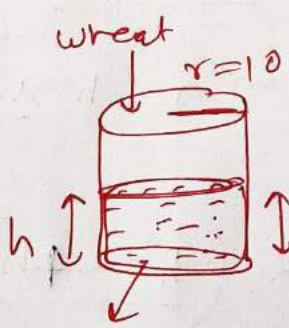
$= \pi \tan^2 \alpha \cdot \left(\frac{2h}{3}\right)^2 \cdot \frac{h}{3}$

$= \frac{4\pi}{27} \tan^2 \alpha \cdot h^3$

[Q.19] A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meter per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h.

- ROC
- D
 - D
 - F
 - D
 - R



$\underline{314 \text{ m}^3/\text{h}} = \frac{dV}{dt}$

$V = \pi r^2 h$
 $V = \pi(100) \cdot h$

$\left(\frac{dV}{dt}\right) = \pi(100) \frac{dh}{dt}$

$\Rightarrow 314 = (3.14)(100) \cdot \left(\frac{dh}{dt}\right)$

$\Rightarrow \frac{dh}{dt} = \frac{314}{314} = \underline{1 \text{ m/h}}$

Miscellaneous Exercise on Chapter 6

Q.20

The slope of the tangent to the curve

 $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

- (A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $-\frac{6}{7}$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t - 2}{2t + 3} = \text{Slope of tangent} = \frac{6}{7}$$

Point $\Rightarrow (2, -1)$

$t = 2$

$$x = t^2 + 3t - 8 = 2$$

$$y = 2t^2 - 2t - 5 = -1$$

$$\Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow t^2 + 5t - 2t - 10 = 0$$

$$\Rightarrow t(t+5) - 2(t+5) = 0$$

$$(t-2)(t+5) = 0$$

$$t = 2, -5$$

$$t = 2$$

$$2t^2 - 2t - 4 = 0$$

$$\Rightarrow t^2 - t - 2 = 0$$

$$\Rightarrow t^2 + t - 2t - 2 = 0$$

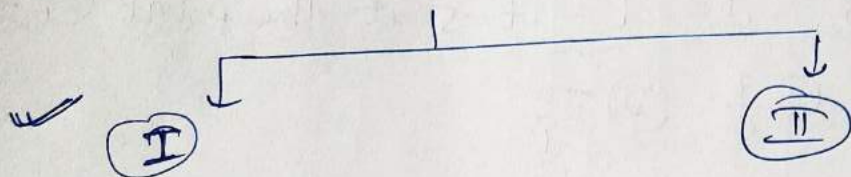
$$\Rightarrow t(t+1) - 2(t+1) = 0$$

$$(t+1)(t-2) = 0$$

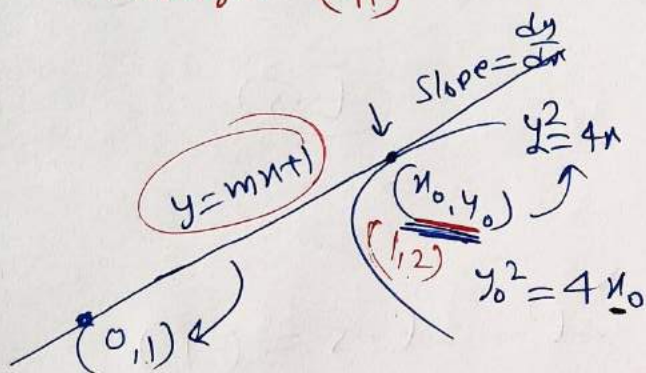
$$t = -1, 2$$

Q.21) The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is -

- (A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$



$y = mx + 1$ always passes through $(0, 1)$



$$y^2 = 4x$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 4$$

Slope of tangent at (x_0, y_0) = $\frac{dy}{dx} = \frac{2}{y} = \frac{2}{y_0} = m$

Tangent: $(y - y_0) = \frac{2}{y_0} (x - x_0)$

Passes through $(0, 1)$ ($\because y_0^2 = 4x_0$)

$$\Rightarrow (1 - y_0) = \frac{2}{y_0} \left(0 - \frac{y_0^2}{4} \right)$$

$$\Rightarrow (1 - y_0) = \frac{-y_0}{2}$$

$$\Rightarrow 2 - 2y_0 = -y_0$$

$$\Rightarrow 2 = y_0$$

$$4 = 4x_0$$

$$x_0 = 1$$

$$(1, 2)$$

$$m = \frac{2}{y_0} = \frac{2}{2} = 1$$

$$y^2 = 4x$$

$$\Rightarrow (mx + 1)^2 = 4x$$

$$\Rightarrow m^2 x^2 + 2mx + 1 = 4x$$

$$\Rightarrow m^2 x^2 + (2m - 4)x + 1 = 0$$

Quadratic

$$D = 0$$

$$b^2 - 4ac = 0$$

$$(2m - 4)^2 - 4m^2 \cdot (1) = 0$$

\Rightarrow Solve

$$m = 1$$

Same roots

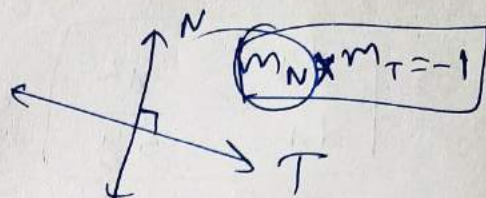
Q.22 The normal at the point $(1,1)$ on the curve $2y+x^2=3$ is —

- (A) $x+y=0$ (B) $x-y=0$ (C) $x+y+1=0$ (D) $x-y=0$

Ans.

$$2 \frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = -x \leftarrow (1,1)$$



Slope of tangent at $(1,1) = m_T = -1$

Slope of normal at $(1,1) = m_N = \frac{-1}{m_T}$

$$m_N = \frac{-1}{-1} = 1$$

Normal:

$$(y-1) = (1)(x-1) \Rightarrow y-1 = x-1$$

$$\Rightarrow \boxed{x-y=0}$$

Q.23 = **Q.4** = **Example 45**

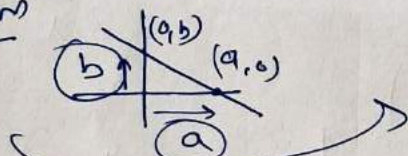
Q.24 The point on the curve $9y^2=x^3$, where the normal to the curve makes equal intercepts with the axes are —

- (A) $(4, \pm \frac{8}{3})$ (B) $(4, -\frac{8}{3})$ (C) $(4, \pm \frac{3}{8})$ (D) $(\pm 4, \frac{3}{8})$

Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\boxed{a=b}$$



Equal intercept

$$\frac{x}{a} + \frac{y}{a} = 1$$

$$\Rightarrow \boxed{x+y=a}$$

$$\boxed{m = -1}$$

$$y = -x + a$$

Curve

$$9y^2 = x^3$$

$$\Rightarrow 9 \times 2y \cdot \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x^2}{6y}}$$

Slope of tangent

m_T →

Slope of normal = $m_N = -\frac{1}{m_T} = -\frac{6y_0}{x_0^2}$

$$m_N = -\frac{6y_0}{x_0^2} = -1$$

$$\Rightarrow 6y_0 = x_0^2 \quad \text{--- (1)}$$

$$\boxed{y_0 = \frac{x_0^2}{6}}$$

$$\boxed{x_0 = 4}$$

$$6y_0 = x_0^2$$

$$6y_0 = 16$$

$$y_0 = \frac{8}{3}$$

$$\boxed{\left(4, \frac{8}{3}\right)}$$

$$9y_0^2 = x_0^3$$

$$9y_0^2 = 64$$

$$y_0^2 = \frac{64}{9}$$

$$y_0 = \pm \frac{8}{3}$$

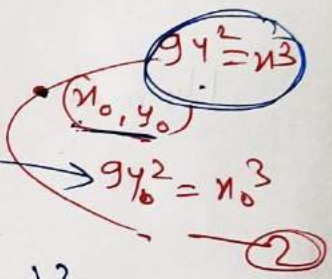
$$\left(4, \pm \frac{8}{3}\right)$$

Equal intercept

$$m = -1$$

$$\frac{x}{a} + \frac{y}{a} = 1$$

Normal makes equal intercepts with axes.



$$9\left(\frac{x_0^2}{6}\right)^2 = x_0^3$$

$$\Rightarrow \frac{9x_0^4}{36} = x_0^3$$

$$\Rightarrow x_0^4 = 4x_0^3$$

$$\Rightarrow x_0^3(x_0 - 4) = 0$$

$$\boxed{x_0 = 0} \quad \boxed{x_0 = 4}$$

$$\boxed{y_0 = 0}$$

$$\left(4, \frac{8}{3}\right) \rightarrow (y - \frac{8}{3}) = -(x - 4) \quad (m = -1)$$

$$\left(4, -\frac{8}{3}\right) \rightarrow (y + \frac{8}{3}) = +(x - 4) \quad (m = +1)$$