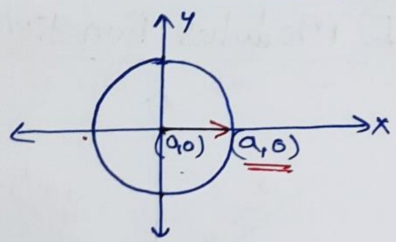
Application of Integrals [समाकलनों के अनुप्रयोग] Area

- Some Graphs
- Some Integration Formulas
- Concept
  - $\int_a^b f(x)\,dx$ → + Area, − Area

Some Graphs:

I Circle

$\boxed{x^2+y^2=a^2}$



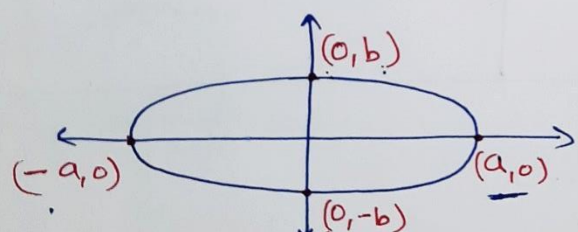
Centre (0,0), radius = a

$\boxed{(x-h)^2+(y-k)^2=a^2}$

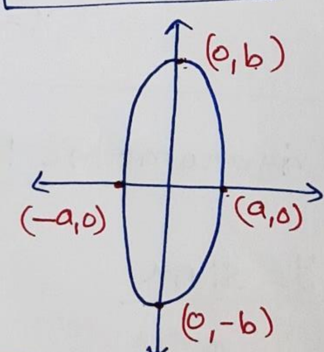
Centre (h,k)
radius = a

II Ellipse

$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $(a>b)$

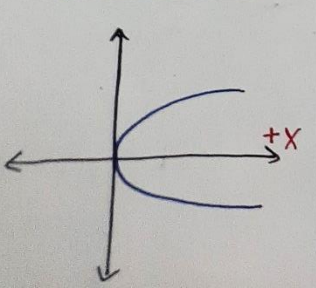


$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $(a<b)$

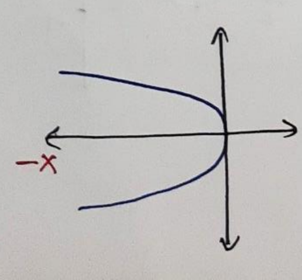


III Parabola

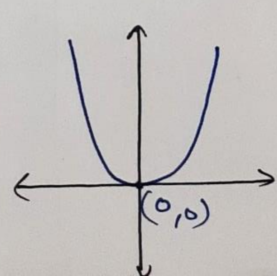
$\boxed{y^2=4ax}$



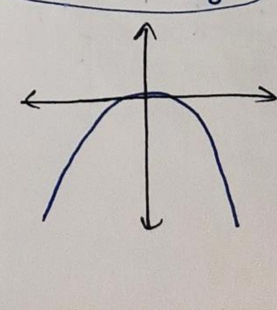
$\boxed{y^2=-4ax}$



$x^2=4ay$



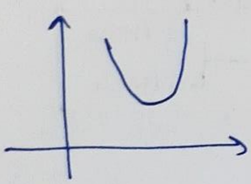
$x^2=-4ay$



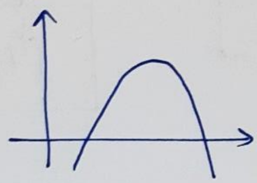
Parabola

$\boxed{y = ax^2 + bx + c}$

$\boxed{a > 0}$ $\boxed{a < 0}$

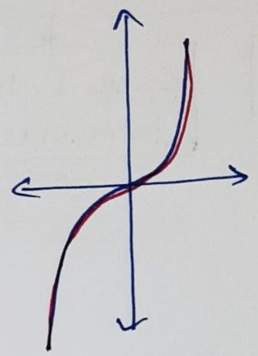


Upward Parabola

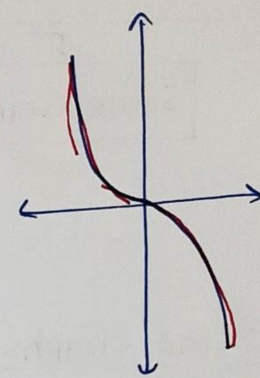


Downward Parabola

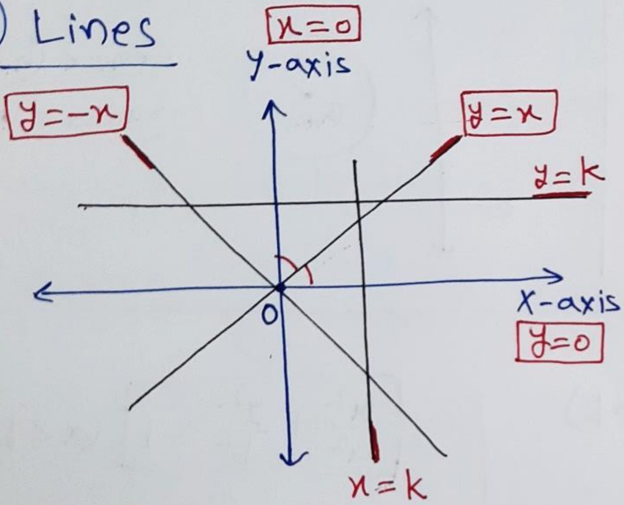
$y = x^3$



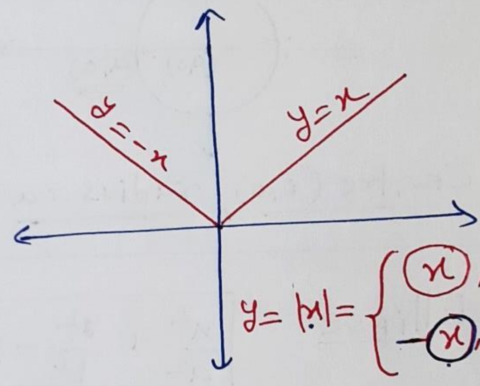
$y = -x^3$



(IV) Lines



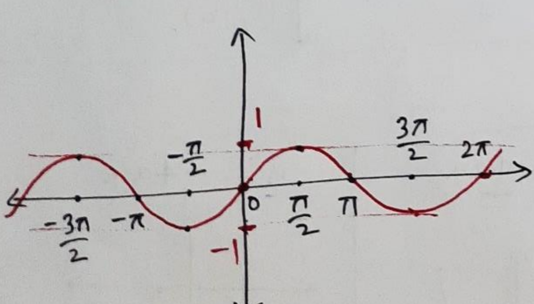
(V) Modulus Function. $\boxed{y=|x|}$



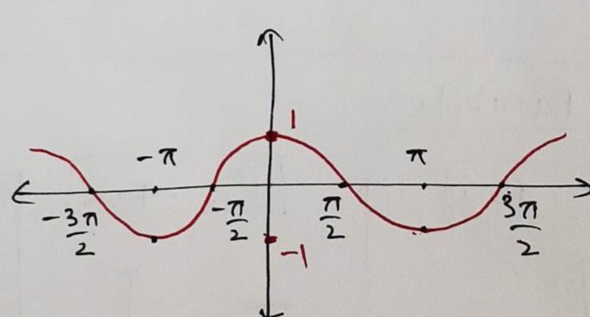
$$y = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

(VI) Trigonometric Functions.

$y = \sin x$



$y = \cos x$



Some Integration Formulas

$\int 1 . dx = x + c$

$\int x^n . dx = \frac{x^{n+1}}{n+1} + c$

$\int x . dx = \frac{x^2}{2} + c$

$\int \sin x . dx = -\cos x + c$

$\int x^2 . dx = \frac{x^3}{3} + c$

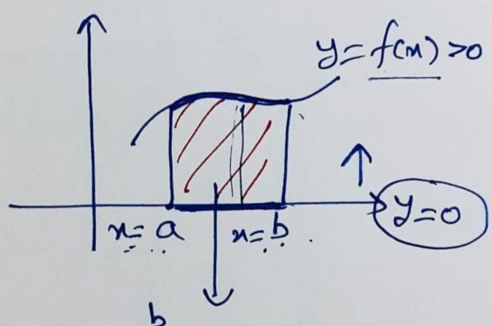
$\int \cos x . dx = \sin x + c$

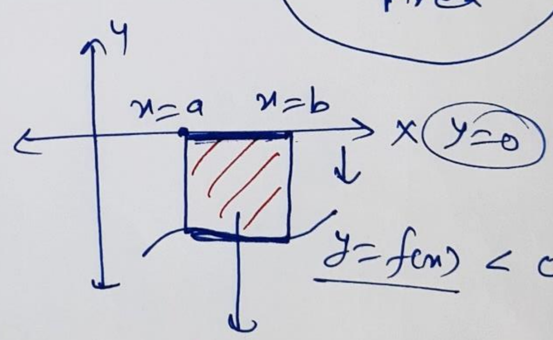
$\int \sqrt{x} . dx = \frac{2}{3} x^{3/2} + c$

$$\boxed{\int \sqrt{a^2 - x^2} . dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

Concept (Application of Integrals)

Meaning of Definite Integrals = Algebraic Area ??





$\text{Area} = \int_a^b f(x) . dx = \oplus ve$

$\text{Area} = \left| \int_a^b f(x) . dx \right| = |\ominus ve| = \oplus$

$$A = \boxed{\int_{x=a}^{x=b} \left(\text{ऊपर वाला Graph} - \text{नीचे वाला Graph} \right) . dx} = \oplus ve \text{ Always}$$

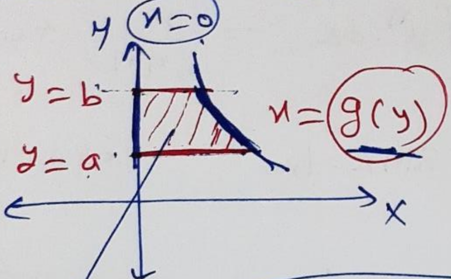
x

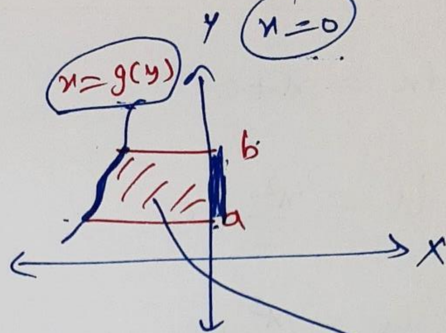
$$\boxed{\text{Area} = \int_a^b (f(x) - 0) . dx = \oplus}$$

$\text{Area} = \int_a^b (0 - f(x)) . dx$

$= \oplus ve$

with respect to 'y'





दाया   बांया

Area = $\int_{y=a}^{y=b}$ (Right वाला − Left वाला) . dy

y की terms.

Area = $\int_a^b \left(g(y) - 0 \right) \cdot dy$

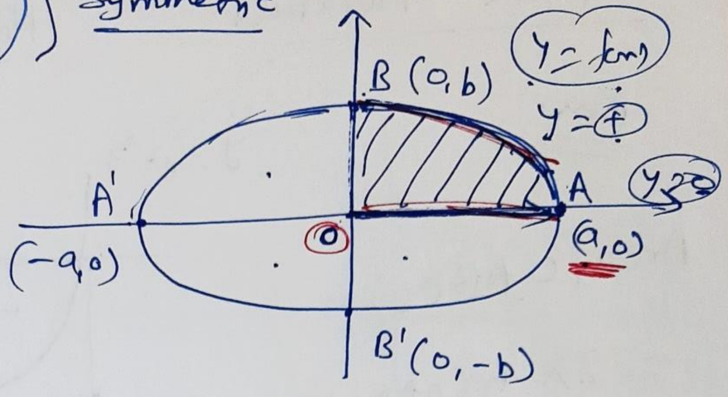
= ⊕ve.

Area = $\int_a^b \left(0 - g(y) \right) \cdot dy$

e.g. Find the area enclosed by the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ans.

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ } Symmetric



Ellipse Area (ABA'B'A)

$= 4\times\boxed{ar(OABO)}$

$= 4\times\int_{x=0}^{x=a}(y-0)\,dx$

Curve   x-axis

(ऊपर − नीचे)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$

$\Rightarrow \frac{y}{b}=\pm\sqrt{\frac{a^2-x^2}{a^2}}$

$\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

+ −

termsin

x

$= 4\times\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)-0-0\right]$ → 0

$= \frac{\overset{2}{\cancel{4}}b}{\cancel{a}}\times\frac{a^{\cancel{2}}}{\cancel{2}}\cdot\sin^{-1}(1)$

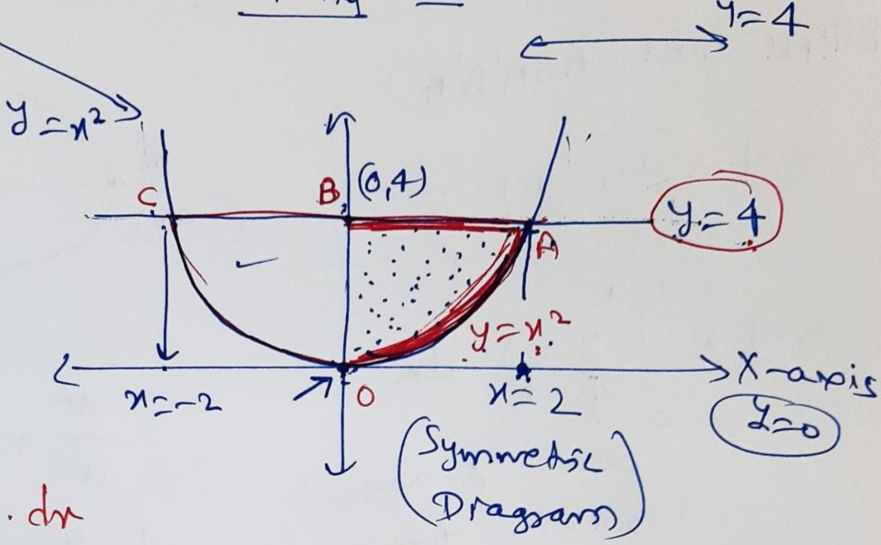
$\sin\frac{\pi}{2}=1$

$= \cancel{2}ba\frac{\pi}{\cancel{2}} = \pi ab$

e.g. Find the area of the region bounded by the curve $\underbrace{y = x^2}$ and line $\underbrace{y = 4}$.

upward Parabola — Horizontal Line

Ans.



Area (OABCO)

$= 2 \times ar(OABO)$

$$= 2 \times \int_{x=0}^{x=2} \underbrace{4}_{\text{line}} - \underbrace{x^2}_{\text{curve}} \cdot dx$$

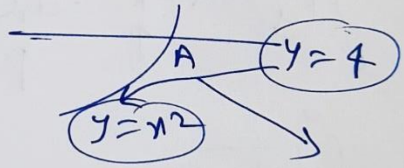
(ऊपर) (नीचे)

$$= 2 \times \left(4x - \frac{x^3}{3}\right)_0^2$$

$$= 2\left(\underbrace{8 - \frac{8}{3}} - \underbrace{0}\right)$$

$$= 2 \times \frac{16}{3} = \frac{32}{3} \text{ ~~Sq~~ unit squares}$$

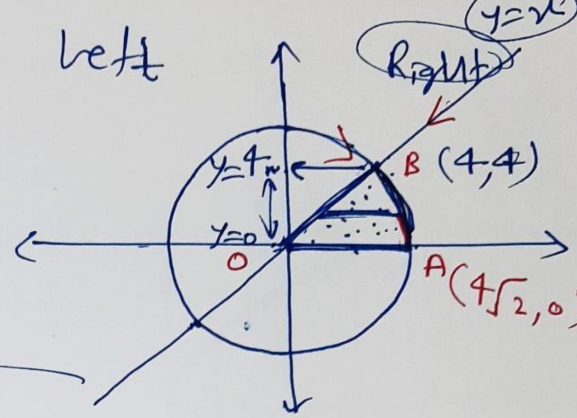
For U.L. = 'A' का X-coordinate



$4 = x^2$

$x = \pm 2$

e.g. Find the area of the region in the first quadrant enclosed by the x-axis, line $y=x$, circle $x^2+y^2=32$.



Ans. Circle $x^2+y^2=32$

Centre $=(0,0)$

$r=\sqrt{32}=\sqrt{16\times 2}=4\sqrt{2}$

$$\int_{y=\text{L.L.}=0}^{y=\text{U.L.}=4}(\text{दाँया}-\text{बाँया})\,dy$$

circle line

$$ar(OABO)=\int_0^4(\text{Curve}-\text{line})\,dy$$

y की terms

$$=\int_0^4\left(\underbrace{\sqrt{32-y^2}}_{x=\text{Curve (दाँया)}}-\underbrace{y}_{\text{line } x=y\ (\text{बाँया})}\right)dy$$

$$=\int_0^4\left(\sqrt{(4\sqrt{2})^2-y^2}-y\right).dy$$

$$=\left[\frac{y}{2}\sqrt{32-y^2}+\frac{32}{2}\sin^{-1}\left(\frac{y}{4\sqrt{2}}\right)-\frac{y^2}{2}\right]_0^4$$

$$=4\pi$$

Point of intersection

B $(4,4)$

$x^2+y^2=32$

$y=x$

$x^2+x^2=32$

$\cancel{2}x^2=\cancel{32}\ 16$

$x=\pm 4$

$x^2+y^2=32$

$x^2=32-y^2$

$x=\pm\sqrt{32-y^2}$

$y=x$

$x=y$

Exercise 8.1

Q.1 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant.

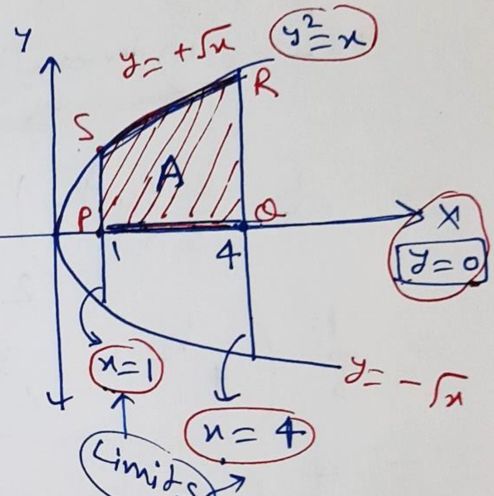
vertical lines.

Ans.

$y = \sqrt{x}$ (curve)   (x-axis)

(ऊपर वाला - नीचे वाला)

$$ar(PQRSP) = \int_{x=1}^{x=4} \underbrace{(\sqrt{x} - 0)}_{\text{in terms of } x} \cdot dx$$



$$= \int_1^4 \sqrt{x} \cdot dx$$

$$= \frac{2}{3}\left(x^{3/2}\right)_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1)$$

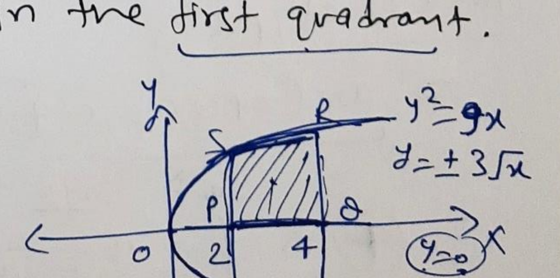
$$= \frac{14}{3} \text{ Sq. units.}$$

Q.2 Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Ans. Vertical

$$ar(PQRSP) = \int_2^4 \underbrace{(3\sqrt{x} - 0)}_{\text{terms in } x} \cdot dx$$

ऊपर   नीचे

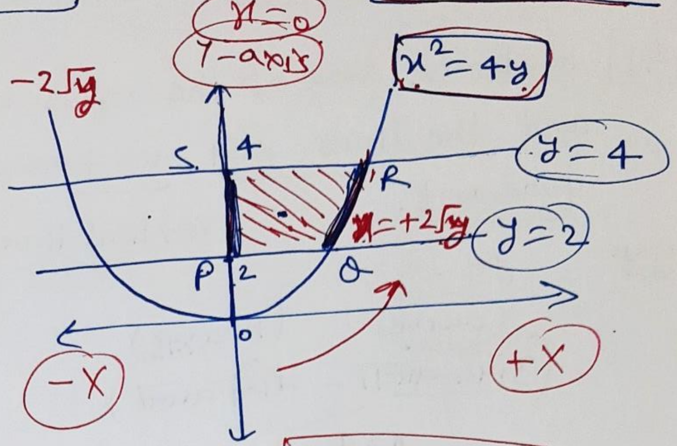


$$= \cancel{3} \cdot \frac{2}{\cancel{3}}\left(x^{3/2}\right)_2^4 = 2\left[4^{3/2} - 2^{3/2}\right]$$

$$= 2(8 - 2\sqrt{2}) = (16 - 4\sqrt{2})$$

Q.3 Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Ans. Horizontal,



ar(PQRSP) (दांया - बांया)

$$= \int_{y=2}^{y=4} (2\sqrt{y} - 0)\,dy$$

in terms of (y)

$$\boxed{\begin{aligned} x^2 &= 4y \\ x &= \pm 2\sqrt{y} \end{aligned}}$$

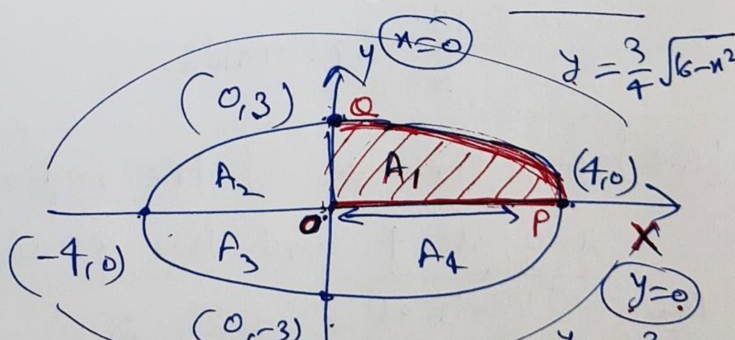
$$= 2 \cdot \frac{2}{3}\left(y^{3/2}\right)_2^4$$

$$= \frac{4}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{4}{3}\left(8 - 2\sqrt{2}\right) = \frac{32 - 8\sqrt{2}}{3} \text{ Sq. units.}$$

Q.4 Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Ans. $A_1 = A_2 = A_3 = A_4$



Ellipse → Symmetrical

Area of ellipse

$= ar(OPQO) \times 4$

(ऊपर - नीचे)

$$= 4 \times \int_0^4 \left(\frac{3}{4}\sqrt{16-x^2} - 0\right).dx$$

terms of (x)

$$\boxed{\begin{aligned} &\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow \frac{y^2}{9} = 1 - \frac{x^2}{16} \\ &\Rightarrow \frac{y}{3} = \pm\sqrt{\frac{16-x^2}{16}} \Rightarrow y = \pm\frac{3}{4}\sqrt{16-x^2} \end{aligned}}$$

$= \cancel{4} \times \frac{3}{\cancel{4}} \int_0^4 \sqrt{16-x^2}\,.\,dx$ $\qquad$ $\underline{a^2 = 16}$ $\qquad$ $a = 4$

$$\boxed{\int \sqrt{a^2-x^2}\,.\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$= 3\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$

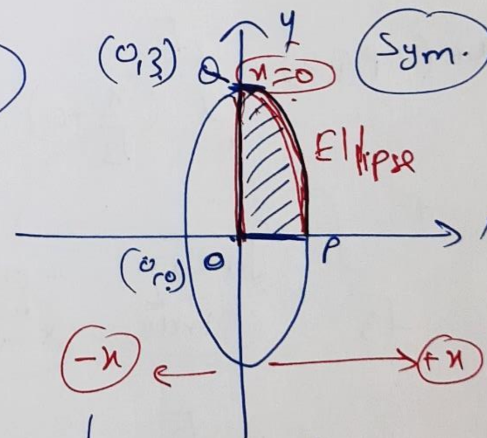
$= 3\left[\underbrace{\frac{4}{2}\cancel{\sqrt{16-16}}}_{0} + 8\sin^{-1}(1) - 0\right]$

$= 3 \times \overset{4}{\cancel{8}} \times \frac{\pi}{\cancel{2}}$

$= 12\pi$

Q.5 Find the area of the region bounded by the ellipse

$\frac{x^2}{4} + \frac{y^2}{9} = 1.$ $\qquad$ $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$



area of ellipse $= 4 \times ar(OPQO)$

$= 4 \times \int_0^3 \left(\frac{2}{3}\sqrt{9-y^2} - 0\right).\,dy$

terms of (y)

(दांया - बांया)

$= \frac{8}{3}\int_0^3 \sqrt{9-y^2}\,.\,dy$

$= \frac{8}{3}\left[\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right]_0^3$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\Rightarrow \frac{x^2}{4} = 1 - \frac{y^2}{9}$

$\Rightarrow x^2 = 4\left(\frac{9-y^2}{9}\right)$

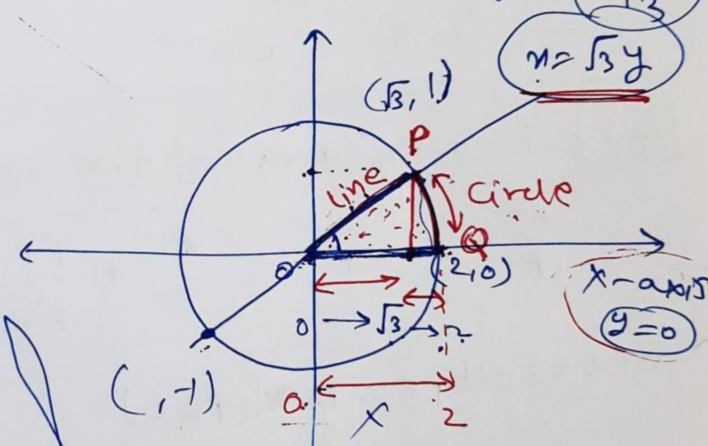
$\Rightarrow x = \pm\frac{2}{3}\sqrt{9-y^2}$ $\quad$ (+)

area of ellipse $= \frac{8}{3}\left(\frac{y}{2}\sqrt{9-y^2}+\frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right]_0^3$

$$= \frac{8}{3}\left(\cancel{\frac{3}{2}\sqrt{9-9}}^{\,0}+\frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)-0\right)$$

$$= \frac{\cancel{8}^{\,4}}{\cancel{3}}\times\frac{\cancel{9}^{\,3}}{\cancel{2}}\sin^{-1}(1) = 12\times\frac{\pi}{2} = 6\pi \text{ sq. units.}$$

[Q.6] Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{3}\,y$ and the circle $x^2+y^2=4$.



Ans. Centre = (0,0)

$r = 2$

$$ar(OPQO) = \int_0^{\sqrt{3}}\left(\frac{x}{\sqrt{3}}-0\right).dx$$

$$+ \int_{\sqrt{3}}^{2}\left(\sqrt{4-x^2}-0\right).dx$$

Circle   x-axis

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}$$

$$+ \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

P (Point of intersection of $x^2+y^2=4$ & $x=\sqrt{3}y$)

$(\sqrt{3}y)^2+y^2=4$

$3y^2+y^2=4$

$4y^2=4 \Rightarrow y^2=1$

$y=\pm 1$

$y=1$

$x=\sqrt{3}$

$$ar(OPQO) = \underbrace{\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\cancel{\sqrt{3}}}\left(\frac{\overset{\sqrt{3}}{\cancel{3}}}{2} - 0\right) + \left[\overset{\to 0}{\cancel{\frac{x}{2}\sqrt{4-4}}} + 2\sin^{-1}(1) - \frac{\sqrt{3}}{2}\sqrt{4-3} - 2\underbrace{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)}\right]$$

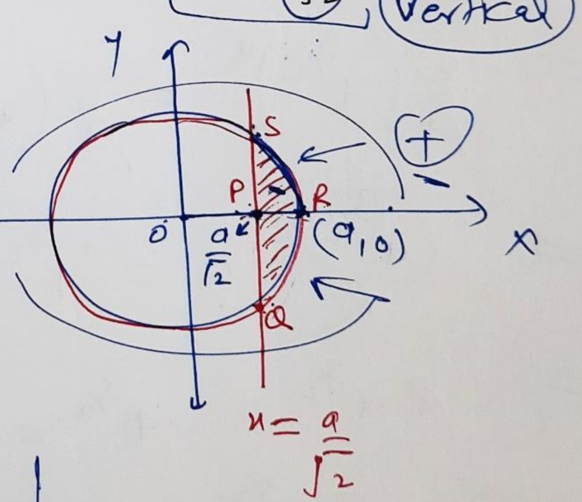
$$= \cancel{\frac{\sqrt{3}}{2}} + \cancel{2}\times\frac{\pi}{\cancel{2}} - \cancel{\frac{\sqrt{3}}{2}} - 2\times\frac{\pi}{3}$$

$$= \pi - \frac{2\pi}{3} = \frac{3\pi - 2\pi}{3} = \frac{\pi}{3}$$

$\boxed{Q.7}$ Find the area of the smaller part of the circle $\underbrace{x^2+y^2=a^2}$ cut off by the line $\underbrace{x = \frac{a}{\sqrt{2}}}$. Vertical

Ans. Centre $(0,0)$, $r = a$

$ar(PRSP) = ar(PRQP)$



$ar(PQRSP) = 2\times \underbrace{ar(PRSP)}$

$$= 2\times\int_{\frac{a}{\sqrt{2}}}^{a}\left(\sqrt{a^2-x^2} - 0\right).dx$$

Circle $y=0$

$x^2+y^2=a^2$

$y = \pm\sqrt{a^2-x^2}$

$$= 2\times\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{\frac{a}{\sqrt{2}}}^{a}$$

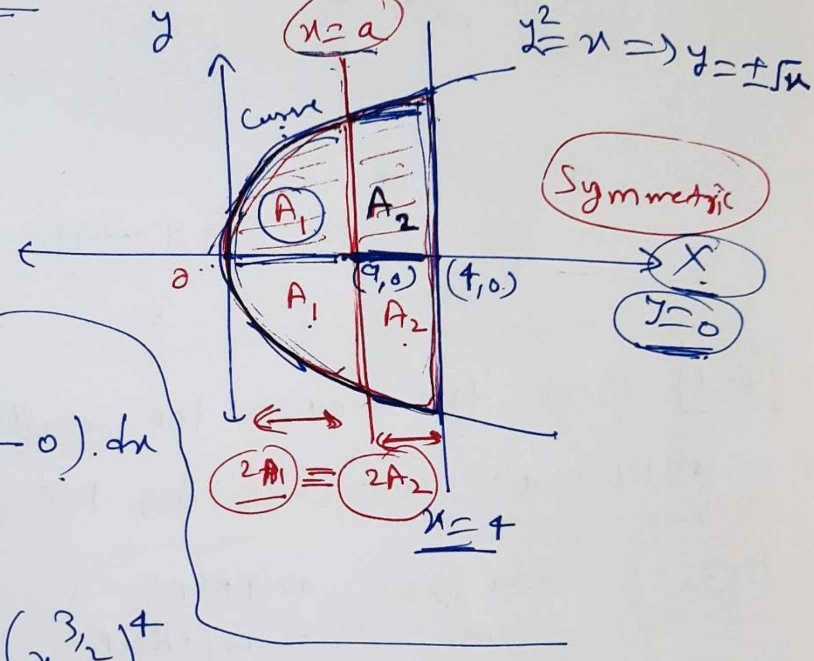
$$= 2\times\left[\overset{\to 0}{\cancel{\frac{a}{2}\sqrt{a^2-a^2}}} + \frac{a^2}{2}\sin^{-1}(1) - \frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

Exercise 8.1

Q.8 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Vertical

Ans.



Given $2A_1 = 2A_2$

$A_1 = A_2$

$$\int_0^a (\sqrt{x} - 0)\,dx = \int_a^4 (\sqrt{x} - 0)\,dx$$

Curve − X Axis Curve

$$\Rightarrow \frac{2}{3}\left(x^{3/2}\right)_0^a = \frac{2}{3}\left(x^{3/2}\right)_a^4$$

$$\Rightarrow a^{3/2} - 0 = 4^{3/2} - a^{3/2}$$

$$\Rightarrow 2a^{3/2} = 4^{3/2} = \left(4^{1/2}\right)^3 = (2)^3 = 8$$

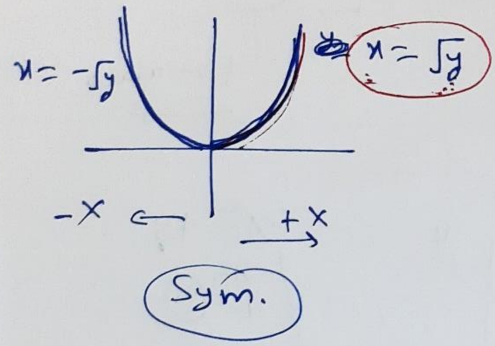
$$\Rightarrow \cancel{2}a^{3/2} = \cancel{8}\,4$$

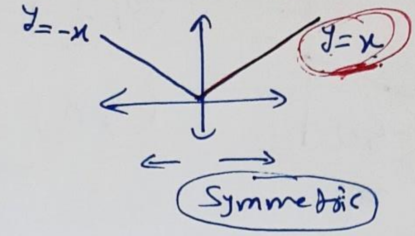
$$\Rightarrow a^{3/2} = 4$$

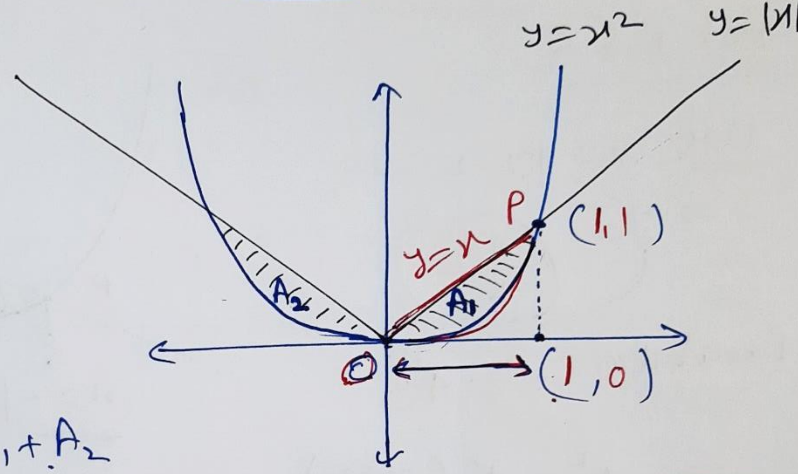
$$\Rightarrow \boxed{a = (4)^{2/3}}$$

$\boxed{Q.9}$ Find the area of the region bounded by the Parabola $y = x^2$ and $y = |x|$.

$\pm\sqrt{y} = x$







By Symmetry

$A_2 = A_1$

Required Area $= A_1 + A_2$

$= 2(A_1)$

terms in (x)

$$= 2\int_{\boxed{0}}^{\boxed{1}} (x - x^2)\, dx$$

(ऊपर) − (नीचे)

Modulus $y = (x)$ Parabola $y = (x^2)$

P $\boxed{y = (x)}$

$\boxed{(x) = \sqrt{y}}$

$y = \sqrt{y}$

$\Rightarrow y^2 = y$

$\Rightarrow y^2 - y = 0$

$y(y-1) = 0$

$y = 0, 1$

$x = 0, 1$

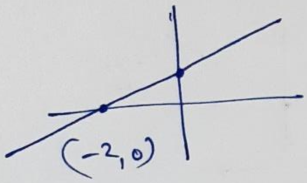
$(0,0)$ $(1,1)$

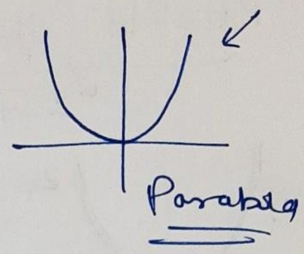
$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1$$

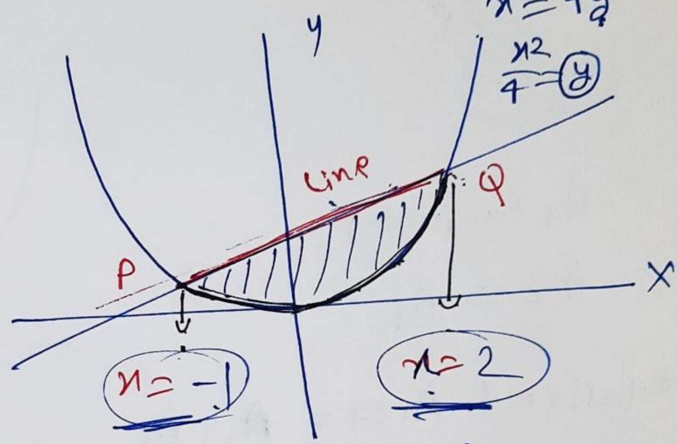
$$= 2\left(\frac{1}{2} - \frac{1}{3} - 0\right)$$

$$= \cancel{2}\left(\frac{3-2}{\cancel{6}_{3}}\right) = \frac{1}{3}$$

Q.10 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.







For P & Q

Line $x = 4y - 2$

$\Rightarrow y = \frac{x+2}{4}$

Parabola: $x^2 = 4y$

$\Rightarrow x^2 = \cancel{4}\left(\frac{x+2}{\cancel{4}}\right)$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow x(x-2) + 1(x-2) = 0$

$\Rightarrow (x-2)(x+1) = 0$

$x = 2, \; x = -1$

Required Area $= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right) dx$

Line   Parabola

$= \frac{1}{4} \int_{-1}^{2} (x + 2 - x^2)\, dx$

$= \frac{1}{4} \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$

$= \frac{1}{4}\left(2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right)$

$= \frac{1}{4}\left(8 - \frac{1}{2} - 3\right)$

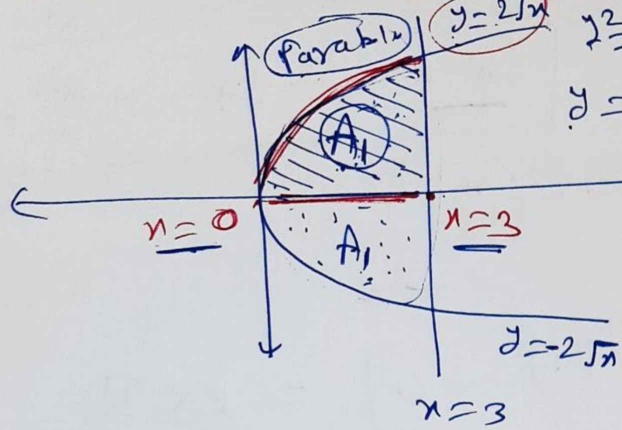
$= \frac{1}{4} \times \frac{9}{2} = \frac{9}{8}$

$5 - \frac{1}{2}$

$\frac{9}{2}$

Q.11 Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Ans.



$y^2 = 4x$

$y = \pm\sqrt{4x} = \pm 2\sqrt{x}$

Symmetric about X-axis

Required Area $= 2(A_1) = 2 \cdot \int_{x=0}^{x=3} (2\sqrt{x} - 0)\, dx$

(3742 - A4) Parabola

$= 4 \int_0^3 \sqrt{x}\, dx$

$= 4 \cdot \frac{2}{3}\left(x^{3/2}\right)_0^3$

$= \frac{8}{3}\left(3^{3/2} - 0\right) = \frac{8}{3} \times 3^{1+\frac{1}{2}} = \frac{8}{\cancel{3}} \times \cancel{3} \cdot 3^{\frac{1}{2}}$

$= 8\sqrt{3}$

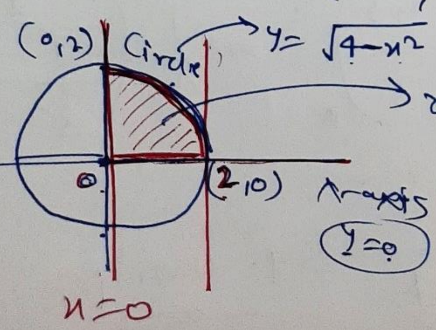
Q.12 Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and lines $x = 0$ & $x = 2$ is —

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Ans.

Centre $(0,0)$, $r = 2$

$y = \sqrt{4-x^2}$



required Area $= \int_{x=0}^{x=2} \left(\sqrt{4-x^2} - 0\right) dx$

3742 - A4 Circle.

required area $= \int_{x=0}^{x=2} \sqrt{4-x^2}\,.dx$ $\quad a^2 = 4,\ a = 2$

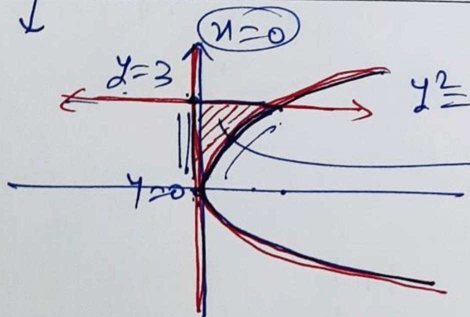
$$\boxed{\int \sqrt{a^2-x^2} = \underbrace{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}}$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{\cancel{4}^{\,2}}{\cancel{2}}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \left[\frac{2}{2}\cancel{\sqrt{4-4}}^{\,0} + \underbrace{2\sin^{-1}(1)} - 0\right] = \cancel{2}\times\frac{\pi}{\cancel{2}} = \pi$$

$\boxed{Q.13}$ Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is —

Ⓐ 2 $\quad$ Ⓑ $\frac{9}{4}$ $\quad$ Ⓒ $\frac{9}{3}$ $\quad$ Ⓓ $\frac{9}{2}$



$y^2 = 4x \Rightarrow \frac{y^2}{4} = x$

The required area

$$= \int_0^3 \left(\frac{y^2}{4} - 0\right).dy$$

Right − Left

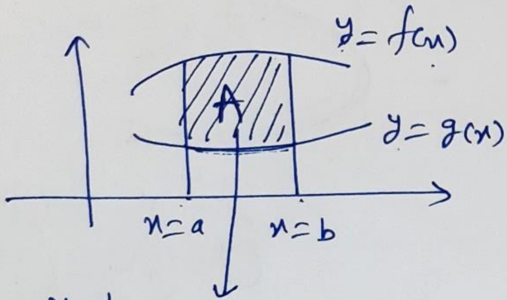
Parabola $\quad$ (y-axis) $x=0$

$$= \int_0^3 \frac{y^2}{4}\,.dy$$

$$= \frac{1}{4}\left(\frac{y^3}{3}\right)_0^3 = \frac{1}{4}\left(\frac{\cancel{27}^{\,9}}{\cancel{3}} - 0\right)$$

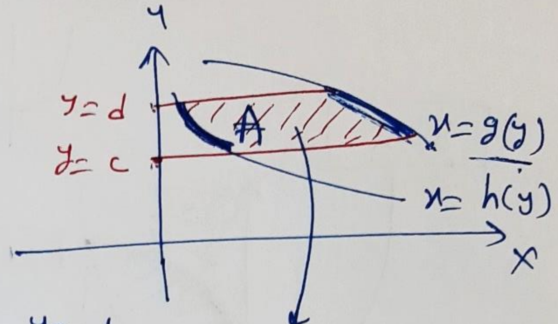
$$= \frac{9}{4} \text{ sq. units.}$$

Area between two Curves (दो वक्रों के मध्य का क्षेत्रफल)



$$\int_{x=a}^{x=b} \left(\text{ऊपर वाला graph} - \text{नीचे वाला graph}\right) . dx$$

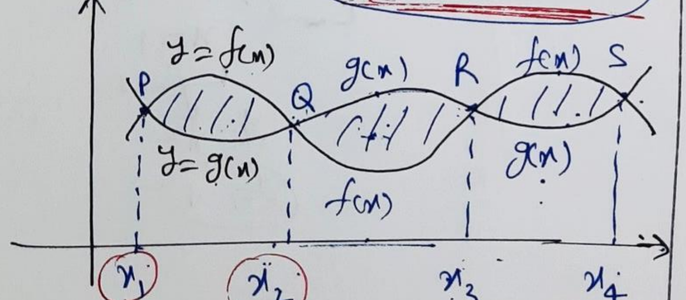
$$\boxed{A = \int_a^b \left(f(x) - g(x)\right) . dx}$$



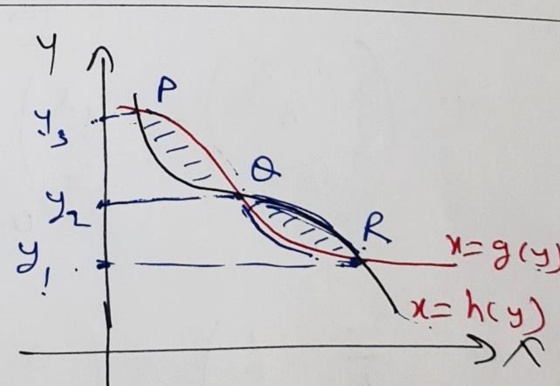
$$\int_{y=c}^{y=d} \left(\text{Right वाला} - \text{left वाला}\right) . dy$$

$$\boxed{A = \int_c^d \left(g(y) - h(y)\right) . dy}$$

Point of intersection



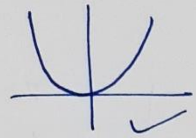
$$A = \int_{x_1}^{x_2} \left(f(x) - g(x)\right) . dx + \int_{x_2}^{x_3} \left(g(x) - f(x)\right) . dx + \int_{x_3}^{x_4} \left(f(x) - g(x)\right) . dx$$

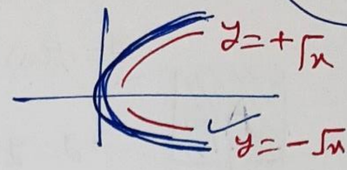


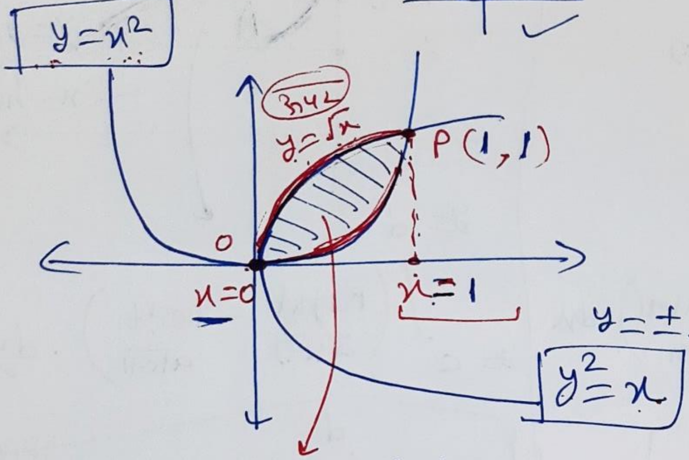
$$A = \int_{y_1}^{y_2} \left(h(y) - g(y)\right) . dy + \int_{y_2}^{y_3} \left(g(y) - h(y)\right) . dy$$

e.g. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

$y = \pm\sqrt{x}$







$y = \pm\sqrt{x}$

$y^2 = x$ — (1)

$y = x^2$ — (2)

For Point 'P' — Solve

$(x^2)^2 = x$

$\Rightarrow x^4 = x$

$\Rightarrow x^4 - x = 0$

$\Rightarrow x(x^3 - 1) = 0$

$x = 0,\ x^3 = 1$

$x = 1$

$y = x^2$

$y = 1$

Required Area $= \displaystyle\int_{x=0}^{x=1} (\sqrt{x} - x^2)\,.\,dx$

terms of x

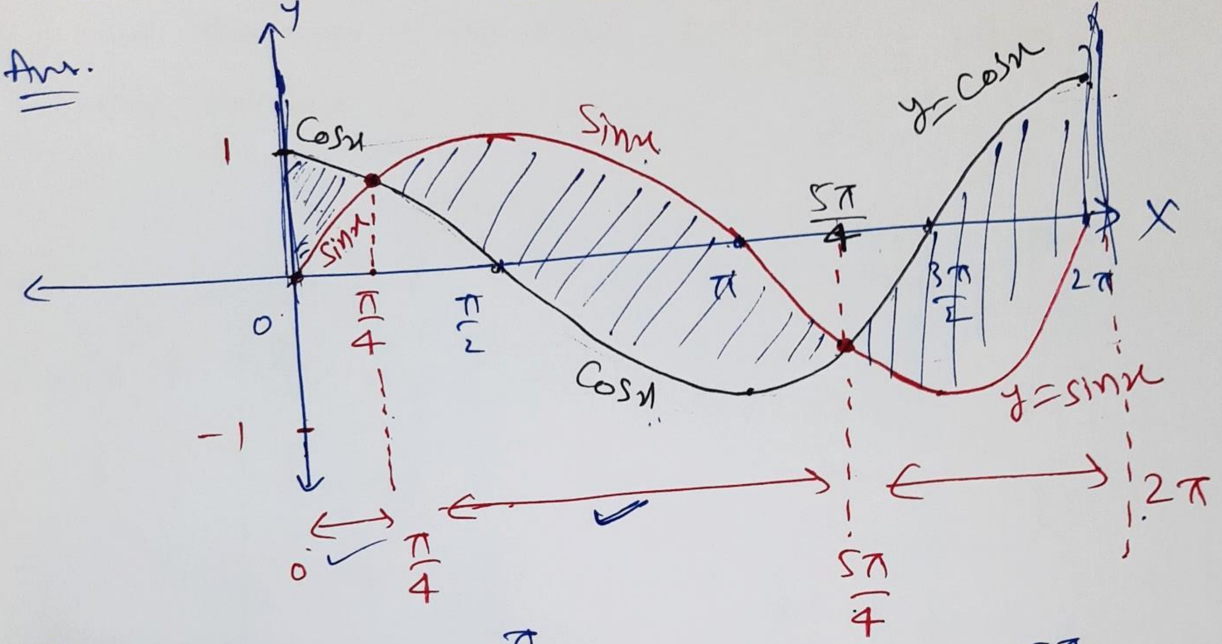
$= \left(\frac{2}{3}\left(x^{3/2}\right) - \frac{x^3}{3}\right)_0^1$

$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0)$

$= \frac{1}{3}$

e.g. The area bounded by the curves $y = \sin x$ and $y = \cos x$ when $0 \le x \le 2\pi$.

Ans.



Required Area $= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,.dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,.dx$

$+ \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,.dx$

$= \left(\sin x + \cos x\right)_0^{\pi/4} + \left(-\cos x - \sin x\right)_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

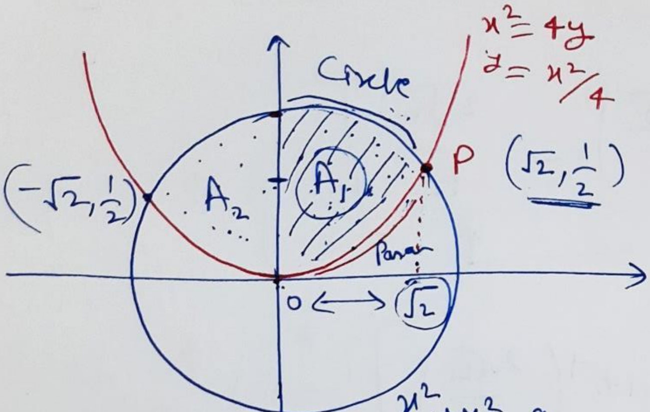
$+ \left(\sin x + \cos x\right)_{\frac{5\pi}{4}}^{2\pi}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - \cancel{1}\right) + \left(+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$+ \left(0 + \cancel{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

Exercise 8.2 (Area between two Curves)

Q.1 Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$.



Symmetric

$A_1=A_2$

Required Area $=A_1+A_2$

$=2A_1$ terms in x

$$=2\int_0^{\sqrt{2}}\left(\sqrt{\frac{9}{4}-x^2}-\frac{x^2}{4}\right).dx$$

(ऊपर - नीचे)

Circle − Parabola

$$=2\int_0^{\sqrt{2}}\left(\sqrt{\frac{9}{4}-x^2}-\frac{x^2}{4}\right).dx$$

$$=2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9/4}{2}\sin^{-1}\left(\frac{x}{3/2}\right)-\frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$x^2+y^2=\frac{9}{4}=\left(\frac{3}{2}\right)^2$$

$$\left[\begin{array}{l}\text{Centre}=(0,0)\\ r=\frac{3}{2}\end{array}\right]$$

For Point P

$x^2=4y$, $4x^2+4y^2=9$

$$4(4y)+4y^2=9$$

$$\Rightarrow 4y^2+16y-9=0$$

$$\Rightarrow 4y^2+18y-2y-9=0$$

$$\Rightarrow 2y(2y+9)-1(2y+9)=0$$

$$\Rightarrow (2y-1)(2y+9)=0$$

$$y=\frac{1}{2},\quad \cancel{y=-\frac{9}{2}}$$

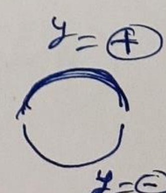
$$x^2=4y=4\left(\frac{1}{2}\right)=2$$

$$x^2=2\Rightarrow \boxed{x=\pm\sqrt{2}}$$

Circle $x^2+y^2=9/4$

$$\Rightarrow y^2=\frac{9}{4}-x^2$$

$$\Rightarrow y=\pm\sqrt{\frac{9}{4}-x^2}$$



Required area $= 2A_1$

$$= 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\underbrace{\sqrt{\frac{9}{4}-2}}_{\frac{1}{2}} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12} - 0\right]$$

$$= 2\left[\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12}\right]$$

LCM

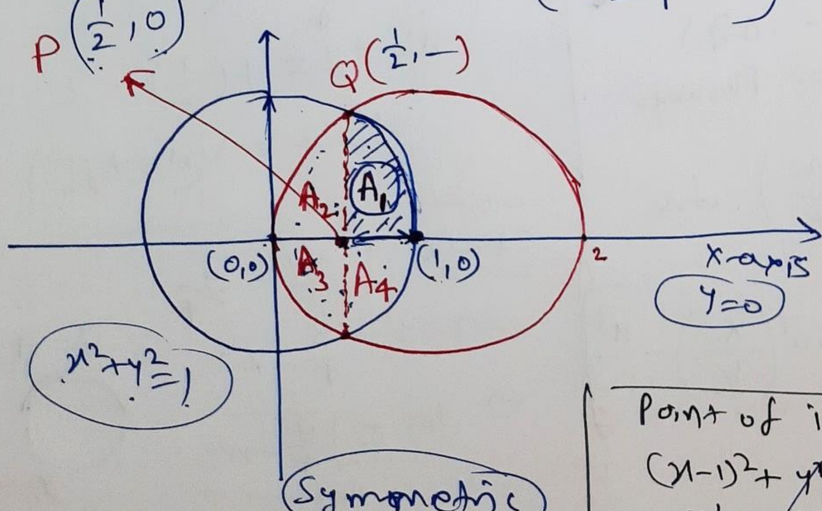
$$= 2\left[\frac{3\sqrt{2}-2\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$\boxed{Q.2}$ Find the area bounded by Curves $\underbrace{(x-1)^2+y^2=1}_{\text{Circle}}$ and $\underbrace{x^2+y^2=1}_{\text{circle}}$.

$x^2+y^2=1$ circle → [centre $(0,0)$, $r=1$]

$(x-1)^2+y^2=1$ Circle → [Centre $(1,0)$, $r=1$]



Required Area

$= A_1 + A_2 + A_3 + A_4$

$= 4A_1$

$A_1 = A_2 = A_3 = A_4$

Point of intersection $Q \to \left(\frac{1}{2},-\right)$

$(x-1)^2 + \cancel{y^2} = \cancel{1}$ →

$-\; x^2 + \cancel{y^2} = \cancel{1}$ →

$\Rightarrow (x-1)^2 - x^2 = 0$ → $x = \frac{1}{2}$

Required Area $= 4A_1$

terms in x

$$= 4 \cdot \int_{x=\frac{1}{2}}^{x=1} \overbrace{\left(\sqrt{1-x^2} - 0\right)}^{\text{terms in } x} \cdot dx$$

(ऊपर - नीचे)

$\downarrow \qquad \searrow$

$x^2+y^2=1 \qquad y=0$

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2}$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{\frac{1}{2}}^{1}$$

$$= 4\left[\underbrace{\cancel{\frac{1}{2}\sqrt{1-1}}}_{0} + \frac{1}{2}\underbrace{\sin^{-1}(1)}_{\pi/2} - \frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}\underbrace{\sin^{-1}\left(\frac{1}{2}\right)}_{\pi/6}\right]$$

$$= 4\left[\frac{\pi}{4} - \frac{1}{4}\sqrt{\frac{3}{4}} - \frac{1}{2}\times\frac{\pi}{6}\right]$$

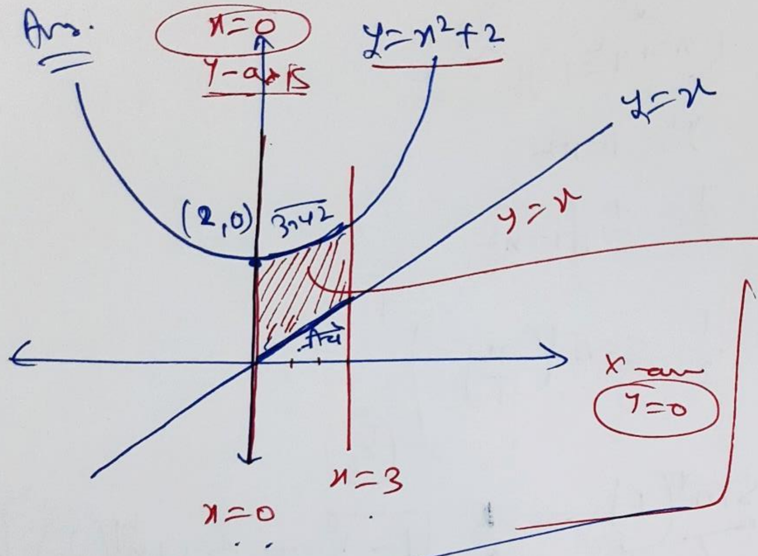
$$= 4\left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right] = 4\left[\frac{3\pi-\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$ ✓

Q.3 Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

$y = x^2+2$ ↑ Parabola; $y = x$ line; $x = 0$ and $x = 3$ vertical lines

Ans.



Required Area

$$= \int_{x=0}^{x=3} \left(\underbrace{x^2+2}_{} - \underbrace{x}_{} \right) dx$$

$y = x^2+2$ $\quad$ $y = x$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2} \right]_0^3$$

$$= \left[9 + 6 - \frac{9}{2} - 0 \right] = \frac{9}{2} + 6 = \frac{9+12}{2} = \frac{21}{2}$$

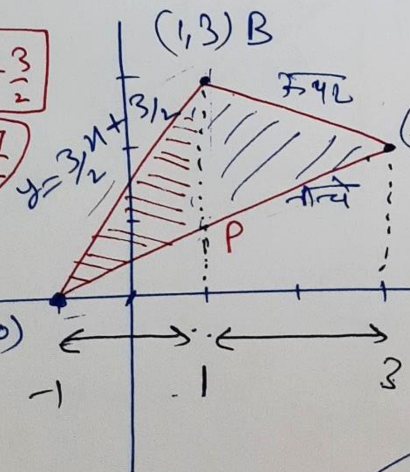
Q.4 Using integration find the area of region bounded by the triangle whose vertices are $(-1,0)$, $(1,3)$ & $(3,2)$

Exactly Plot.

AB $\boxed{y = \frac{3}{2}x + \frac{3}{2}}$

BC $\boxed{y = -\frac{x}{2} + \frac{7}{2}}$

AC $\boxed{y = \frac{x}{2} + \frac{1}{2}}$



Equation (x_1, y_1), (x_2, y_2)

$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)(x - x_1)$$

AB $\quad y - 3 = \left(\frac{3-0}{1+1} \right)(x - 1)$

$$\Rightarrow y = \frac{3}{2}x - \frac{3}{2} + 3$$

$$\Rightarrow \boxed{y = 3/2\, x + 3/2}$$

Required Area = ar(Δ APB) + ar(Δ BPC)

$$= \int_{-1}^{1} \left(\frac{3}{2}x + \frac{3}{2} - \frac{x}{2} - \frac{1}{2}\right) dx + \int_{1}^{3} \left(-\frac{x}{2} + \frac{7}{2} - \frac{x}{2} - \frac{1}{2}\right) dx$$

(AB − AP) $\qquad$ BC − PC

$$= \int_{-1}^{1} (x+1)\, dx + \int_{1}^{3} (-x+3)\, dx$$

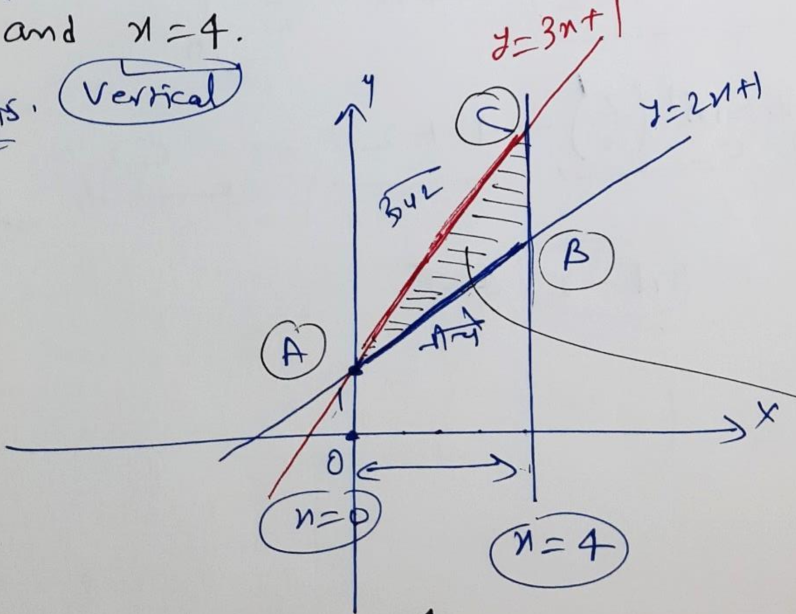
$$= \left(\frac{x^2}{2} + x\right)_{-1}^{1} + \left(-\frac{x^2}{2} + 3x\right)_{1}^{3}$$

$$= \left[\cancel{\frac{1}{2}} + 1 - \cancel{\frac{1}{2}} + 1\right] + \left[-\frac{9}{2} + 9 + \frac{1}{2} - 3\right]$$

$$= 8 - 4 = 4$$

[Q.5] Using Integration find the area of the triangle region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

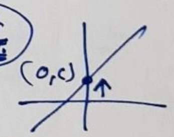
Ans. Vertical



$y = mx + c$

$m_1 < m_2$

1



Required area = ar(ΔABC) $= \int_{0}^{4} (3x + 1 - 2x - 1)\, dx$

$$= \int_{0}^{4} x\, dx = \left(\frac{x^2}{2}\right)_0^4 = \frac{16}{2} - 0 = 8$$

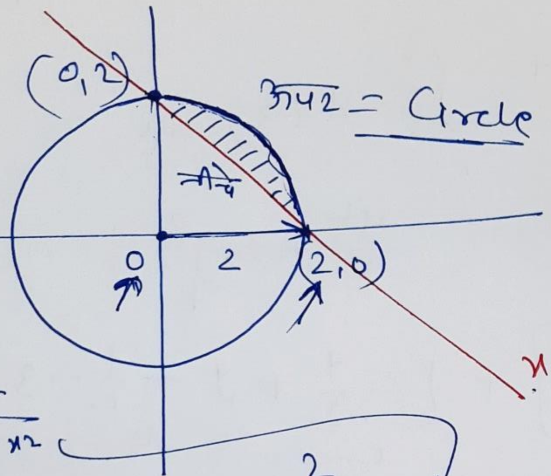
Q.6 Smaller area enclosed by the circle $x^2+y^2=4$ and the line $x+y=2$, is —

(A) $2(\pi-2)$ (B) $\pi-2$ (C) $2\pi-1$ (D) $2(\pi+2)$

Ans.

$x^2+y^2=4=r^2$

(Centre $(0,0)$, $r=2$)



$x^2+y^2=4$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$y=2-x$

Required area $= \int_0^2 \left(\sqrt{4-x^2}-2+x\right).dx$

ऊपर − नीचे

$= \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-2x+\frac{x^2}{2}\right]_0^2$

$= \left[\cancel{\frac{2}{2}\sqrt{4-4}}^{\,0}+2\sin^{-1}\left(\frac{2}{2}\right)-4+2-0\right]$

$= \cancel{2}\times\frac{\pi}{\cancel{2}}-2 = \pi-2$

Q.7 Area lying between the curves $y^2 = 4x$ and $y = 2x$ is — (A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

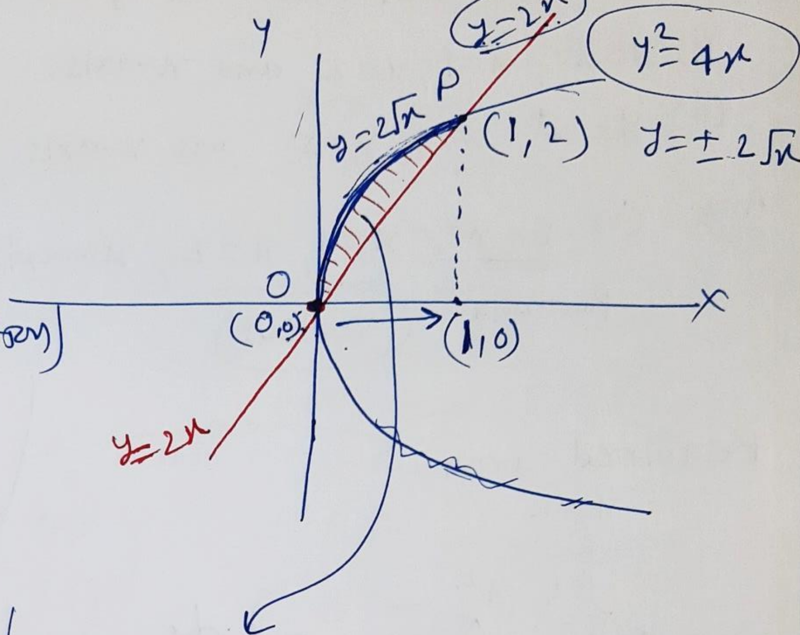
Ans.

$y^2 = 4x$ (Parabola)

~~D~~ C

$y = 2x$

$y = mx + c$



P (Point of intersection)

$y = 2x$ —

$y^2 = 4x$ —

$\Rightarrow (2x)^2 = 4x$

$\Rightarrow \cancel{4}x^2 = \cancel{4}x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$x = 0, x = 1$

O P

$y = 2(x)$

$y = 2$

P(1,2)

Required Area

$$= \int_0^1 (2\sqrt{x} - 2x)\,dx$$

(ऊपर – नीचे)

$y = 2\sqrt{x}$ $y = 2x$

$$= \left(2 \cdot \frac{2}{3} \cdot x^{3/2} - \cancel{2} \cdot \frac{x^2}{\cancel{2}}\right)_0^1$$

$$= \left(\frac{4}{3} \times 1 - 1 - 0\right) = \frac{4}{3} - 1 = \frac{4-3}{3}$$

$$= \frac{1}{3}$$

Miscellaneous Exercise on Chapter 8 ⇒

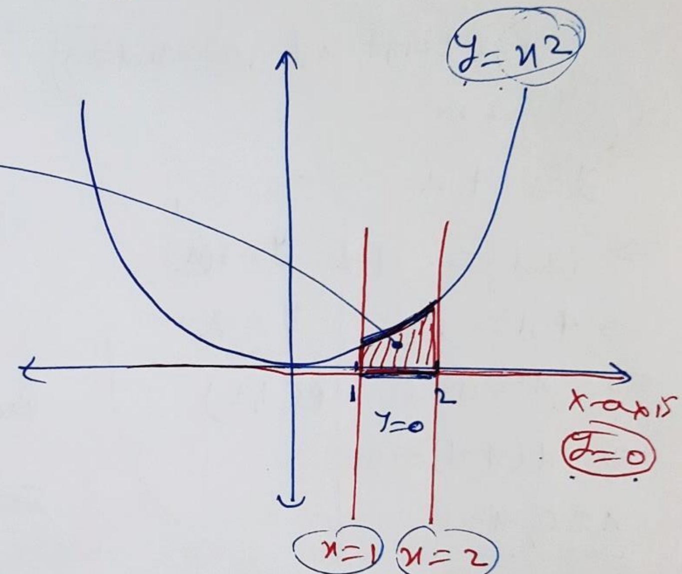
Q.1 Find the area under the given curves and given lines :

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis.

(ii) $y = x^4$, $x = 1$, $x = 5$ and x-axis

Ans. (i) $y = x^2$, $x = 1$, $x = 2$, x-axis

Parabola   vertical



Required area

$$= \int_{x=1}^{x=2} (x^2 - 0)\,dx$$

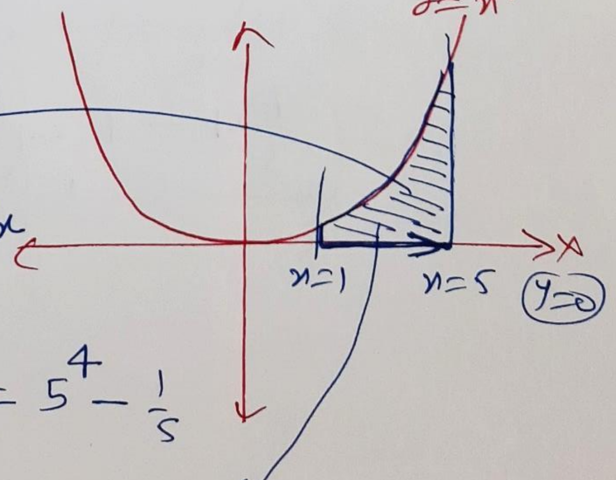
ऊपर − नीचे

$y = x^2$ $y = 0$

$$= \left(\frac{x^3}{3}\right)_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(ii) $y = x^4$, $x = 1$, $x = 5$, x-axis

vertical



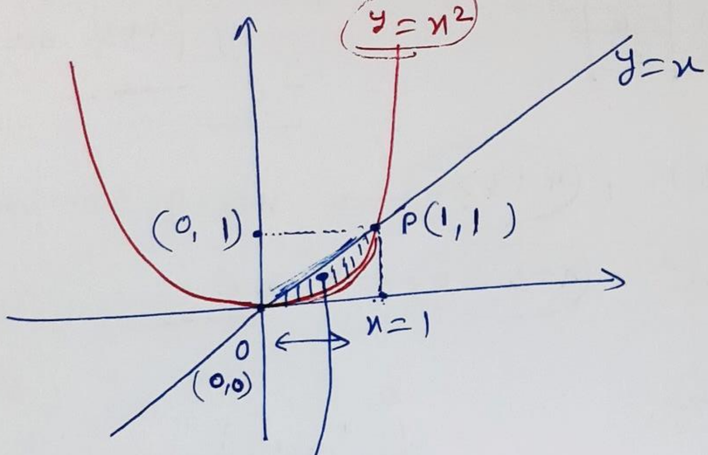
$$\text{Required area} = \int_{x=1}^{x=5} (x^4 - 0)\,dx$$

ऊपर − नीचे

$$= \left(\frac{x^5}{5}\right)_1^5 = \frac{5^5}{5} - \frac{1}{5} = 5^4 - \frac{1}{5}$$

$$= 625 - 0.2 = 624.8$$

Q.2 Find the area between the curves $y = x$ and $y = x^2$.



For 'P' (Point of intersection)

$y = x$ — (1)

$y = x^2$ — (2)

$\Rightarrow x = x^2$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$, $\boxed{x = 1}$ $\boxed{y = 1}$

P

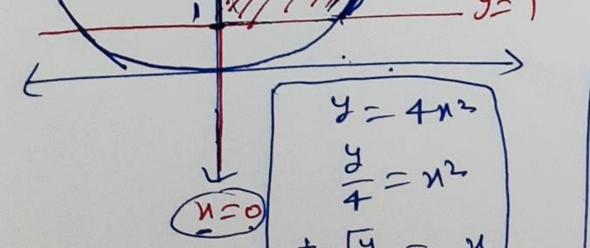
$$\text{Required Area} = \int_{x=0}^{x=1} (x - x^2)\,dx$$

$y = x$ $\qquad$ $y = x^2$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0) = \frac{3-2}{6} = \frac{1}{6}$$

Q.3 Find the area of the region lying in the first quadrant and bounded by $\underbrace{y = 4x^2}_{\text{Parabola}}$, $\underbrace{x = 0}_{Y\text{-axis}}$, $\underbrace{y = 1, y = 4}_{\text{Horizontal}}$.



$$y = 4x^2$$
$$\frac{y}{4} = x^2$$
$$\pm \frac{\sqrt{y}}{2} = x$$

Required Area

$$= \int_{y=1}^{y=4} \left(\frac{\sqrt{y}}{2} - 0\right).dy$$

Right − Left

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \left(y^{3/2}\right)_1^4$$

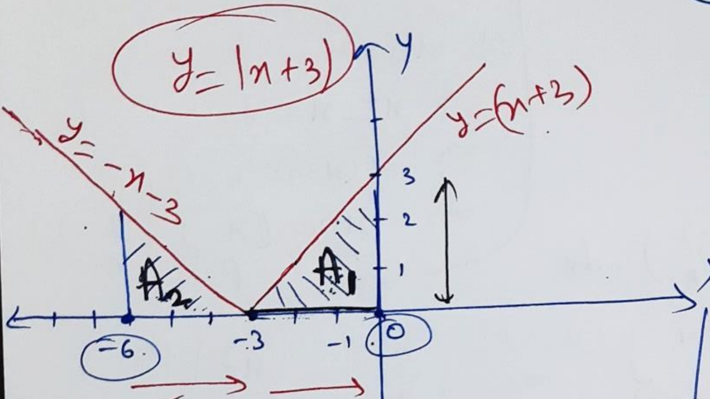
$$= \frac{1}{3}\left[4^{3/2} - 1\right] = \frac{1}{3}(8 - 1)$$

$$= 7/3$$

Q.4 Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3| \cdot dx$

x	0	1	-1	-2	-3	-4	-5	-6
y	3	4	2	1	0	1	2	3

$$y = |x+3| = \begin{cases} +(x+3), & x+3 \geq 0 \rightarrow x \geq -3 \\ -(x+3), & x+3 < 0 \rightarrow x < -3 \end{cases}$$



$$\int_{-6}^{0} |x+3| \cdot dx$$

$x+3 = 0$

$x = -3$

$$= \int_{-6}^{-3} (-x-3) \cdot dx + \int_{-3}^{0} (x+3) \cdot dx$$

$= 9$

$\int_{-6}^{0} |x+3| \cdot dx$ = Area under the curve $y = (x+3)$ upto x-axis

$A_1 = A_2$

$= A_2 + A_1$

$= 2 A_1$

$A_1 = ar(\Delta)$

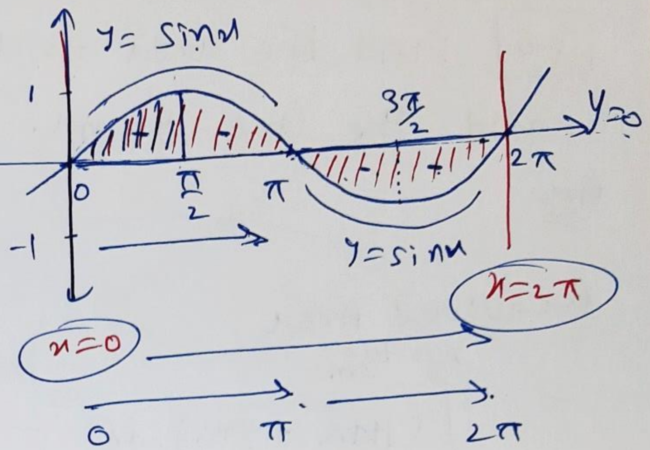
$= \cancel{2} \times \left(\frac{1}{\cancel{2}} \cdot \text{Base} \times \text{Height}\right)$

$= 3 \times 3$

$= 9$ a

Q.5 Find the area bounded by the curve $y = \sin x$ between $x = 0$ & $x = 2\pi$.

Ans



Required area

$$= \int_0^{\pi} (\sin x - 0)\,.dx + \int_{\pi}^{2\pi} (0 - \sin x)\,.dx$$

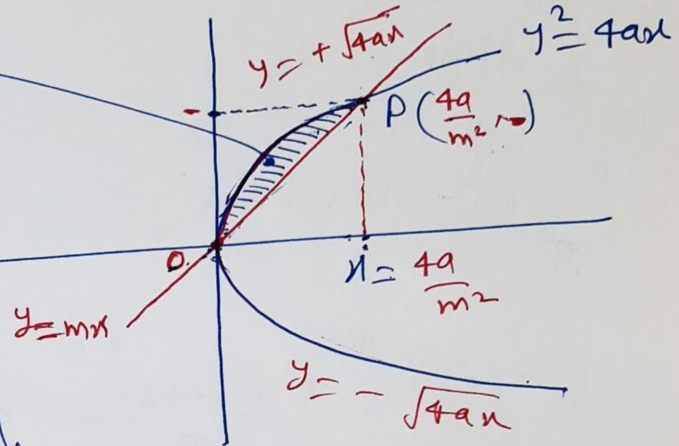
$$= (-\cos x)_0^{\pi} + (\cos x)_{\pi}^{2\pi} = (-\cos\pi + \cos 0) + (\cos 2\pi - \cos\pi)$$

$$= 1 + 1 + 1 + 1 = 4$$

Miscellaneous Exercise on Chapter 8

Q.6 Find the area enclosed by parabola $y^2 = 4ax$ and the line $y = mx$.

Ans.



Required Area

$$= \int_{x=0}^{x=4a/m^2} \left(\sqrt{4ax} - mx\right) dx$$

$(\text{Parabola} - \text{line})$

Parabola: $y^2 = 4ax$, $y = \pm\sqrt{4ax}$

line: $y = mx$

P (Point of intersection)

$y = mx \quad | \quad y^2 = 4ax$

$\Rightarrow (mx)^2 = 4ax$

$\Rightarrow m^2x^2 - 4ax = 0$

$\Rightarrow x(m^2x - 4a) = 0$

$\Rightarrow x = 0, \; x = \frac{4a}{m^2}$

$$= \int_0^{4a/m^2} \left(2\sqrt{a}\sqrt{x} - mx\right) dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{3/2} - m\frac{x^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2\right] - [0]$$

$$= \frac{4}{3}\sqrt{a}\cdot\frac{8a\sqrt{a}}{m^3} - \frac{\cancel{m}}{\cancel{2}}\cdot\frac{\overset{8}{\cancel{16}}a^2}{\cancel{m^4}\,m^3} = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

$$= \frac{32a^2 - 24a^2}{3m^3} = \boxed{\frac{8a^2}{3m^3}}$$

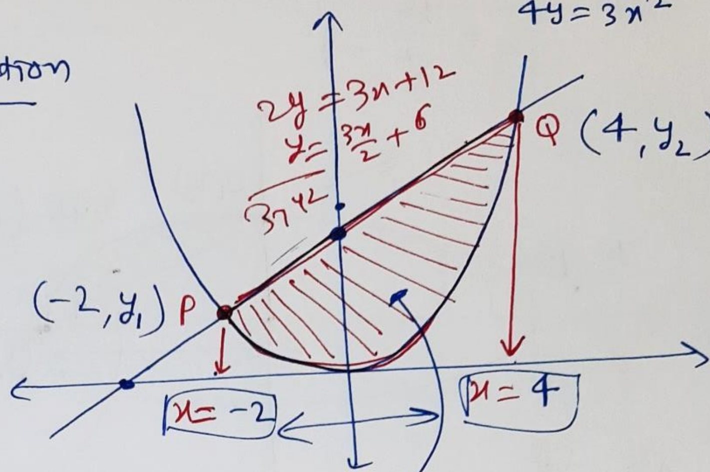
$\boxed{Q.7}$ Find the area enclosed by the Parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

$y = \frac{3}{4}x^2$

$4y = 3x^2$

Ans.

P & Q → point of intersection



$2y = 3x + 12$

$4y = 3x^2$

$\Rightarrow 2(2y) = 3x^2$

$\Rightarrow 2(3x + 12) = 3x^2$

$\Rightarrow 6x + 24 = 3x^2$

$\Rightarrow \frac{3x^2 - 6x - 24}{3} = \frac{0}{3}$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x^2 - 4x + 2x - 8 = 0$

$\Rightarrow x(x-4) + 2(x-4) = 0$

$(x+2)(x-4) = 0$

$\boxed{x = -2}$ $\boxed{x = 4}$

Required area $= \int_{-2}^{4} \left(\frac{3x}{2} + 6 - \frac{3}{4}x^2\right) . dx$

ऊपर − नीचे

Line Parabola

$= \left[\frac{3}{2}\frac{x^2}{2} + 6x - \frac{3}{4}\cdot\frac{x^3}{\cancel{3}}\right]_{-2}^{4}$

$= \left(\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right)_{-2}^{4}$

$= \left(\frac{3}{\cancel{4}}\cdot\cancel{16}^{4} + 24 - 16\right) - \left(3 - 12 + 2\right)$

$= 20 + 7 = 27$

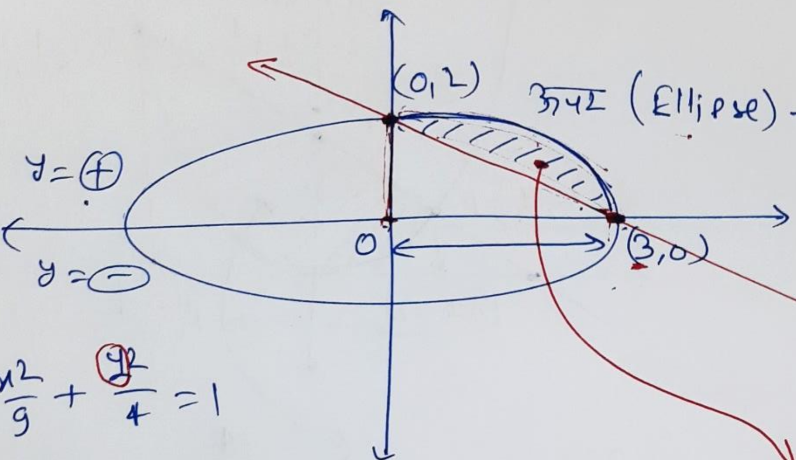
Q.8 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

Intercept.

$\frac{x}{a}+\frac{y}{b}=1$

Ans.

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$



अपर (Ellipse) $\rightarrow y=\frac{2}{3}\sqrt{9-x^2}$

$\frac{x}{3}+\frac{y}{2}=1$

$\frac{y}{2}=1-\frac{x}{3}$

Line $\boxed{y=2-\frac{2x}{3}}$

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\Rightarrow \frac{y^2}{4}=1-\frac{x^2}{9}$$

$$\Rightarrow y^2=4\left(\frac{9-x^2}{9}\right)$$

$$\Rightarrow \boxed{y=\pm\frac{2}{3}\sqrt{9-x^2}}$$

Required Area

$$=\int_0^3\left(\frac{2}{3}\sqrt{9-x^2}-2+\frac{2x}{3}\right).dx$$

अपर − नीचे (Line)

$$=\left[\frac{2}{3}\left\{\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right\}-2x+\frac{2}{3}\frac{x^2}{2}\right]_0^3$$

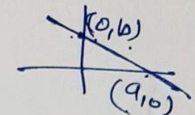
$$=\left[\frac{2}{3}\left\{\frac{3}{2}\cancel{\sqrt{9-9}}^{\,0}+\frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\}-6+3\right]-[0]$$

$$=3\sin^{-1}(1)-3$$

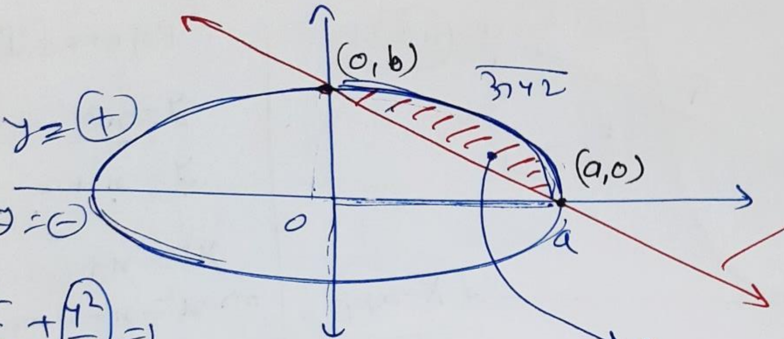
$$=3\left(\frac{\pi}{2}\right)-3$$

$$=3\left(\frac{\pi}{2}-1\right)=3\left(\frac{\pi-2}{2}\right)=\frac{3}{2}(\pi-2)$$

Q.9 Find the area of smaller region bounded by the ellipse $\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}_{\text{Standard Form.}}$ and the line $\underbrace{\frac{x}{a}+\frac{y}{b}=1}_{\text{Intercept Form}}$.



Ans.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\Rightarrow y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$\Rightarrow \boxed{y=\pm\frac{b}{a}\sqrt{a^2-x^2}}$$

$$\frac{x}{a}+\frac{y}{b}=1$$

$$y=\left(b-\frac{bx}{a}\right)$$

Required Area

$$=\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}-b+\frac{bx}{a}\right).dx$$

(उपर − नीचे)

$$=\left[\frac{b}{a}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\}-bx+\frac{bx^2}{a2}\right]_0^a$$

$$=\left[\frac{b}{a}\left\{\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}(1)\right\}-ba+\frac{ba2}{2a}\right]-[0]$$

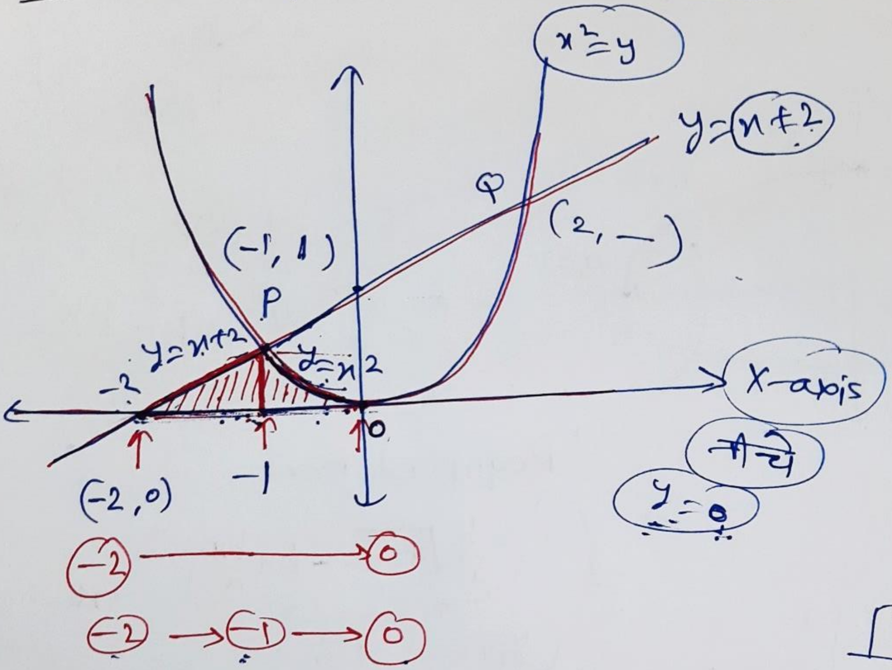
$$=\frac{ab}{2}\cdot\frac{\pi}{2}-ba+\frac{ba}{2}=\frac{\pi ab}{4}-\frac{ba}{2}$$

$$=ab\left\{\frac{\pi}{4}-\frac{1}{2}\right\}=ab\left\{\frac{\pi-2}{4}\right\}=\frac{ab}{4}(\pi-2)$$

Q.10 Find the area of the region enclosed by the Parabola $x^2 = y$, the line $y = x+2$ and the x-axis.

$y = mx + c$

Ans.



For P (Point of Intersection)

$y = x^2$ — ①

$y = x + 2$ — ②

$x^2 = x + 2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x-2)(x+1) = 0$

$\boxed{x = 2, -1}$

Required Area $= \int_{-2}^{-1} (x+2 - 0).dx + \int_{-1}^{0} (x^2 - 0).dx$

$= \left(\frac{x^2}{2} + 2x\right)_{-2}^{-1} + \left(\frac{x^3}{3}\right)_{-1}^{0}$

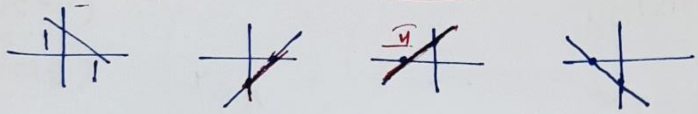
$= \left(\frac{1}{2} - 2\right) - (2 - 4) + (0) - \left(\frac{-1}{3}\right)$

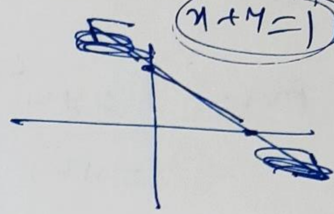
$= \frac{1}{2} - \cancel{2} + \cancel{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}$

Miscellaneous Exercise on Chapter 8

Q.11 Using the method of integration find the area bounded by the curve $|x|+|y|=1$. [Hint: The required region is bounded by lines $x+y=1$, $x-y=1$, $-x+y=1$ and $-x-y=1$]

Ans.

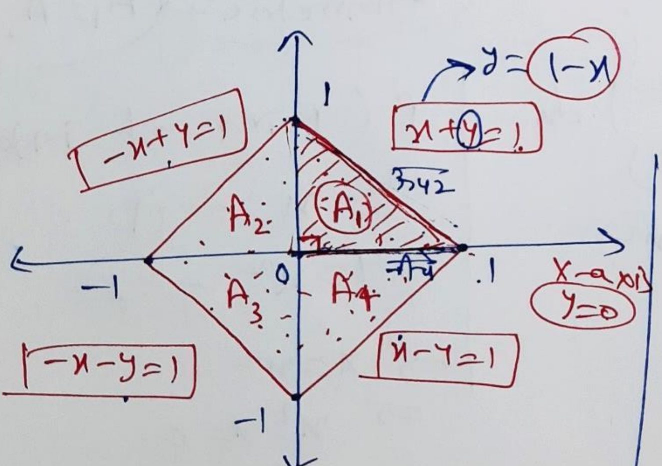




$\boxed{|x|+|y|=1}$

Quadrant wise

	(x, y)	
I-Quad. $\longrightarrow$	$(+,+)$ $\longrightarrow$	$\boxed{x+y=1}$
II-Quad. $\longrightarrow$	$(-,+)$ $\longrightarrow$	$\boxed{-x+y=1}$
III-Quad. $\longrightarrow$	$(-,-)$ $\longrightarrow$	$\boxed{-x-y=1}$
IV-Quad. $\longrightarrow$	$(+,-)$ $\longrightarrow$	$\boxed{x-y=1}$



Symmetric
$(A_1 = A_2 = A_3 = A_4)$

Required Area

$= A_1 + A_2 + A_3 + A_4$

$= 4(A_1)$

$= 4 \cdot \int_{x=0}^{x=1} \big(\underbrace{(1-x)}_{\text{ऊपर}} - \underbrace{0}_{\text{नीचे}} \big) \cdot dx$

$= 4 \int_0^1 (1-x) \cdot dx$

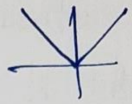
$= 4 \left(x - \dfrac{x^2}{2} \right)_0^1$

$= 4\left(\underbrace{1 - \tfrac{1}{2}} - 0 \right) = 4 \times \tfrac{1}{2}$

$= 2$ Sq. units.

Q.12 Find the area bounded by curves

$\{(x, y) : y \geq x^2$ and $y = |x|\}$.



$y = x^2$ → Graph

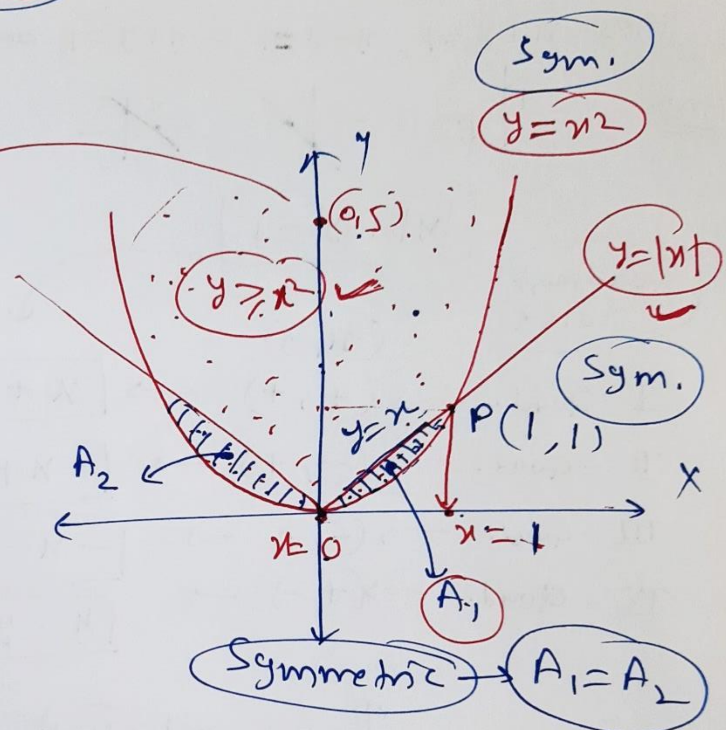
Hit & Trial. → Point Satisfy

Ans. $y \geq x^2$ ← $(0, 5)$

$y = x^2$

$\boxed{5 \geq 0}$

True



Required Area

$= A_1 + A_2$

$= 2(A_1)$

$$= 2 \cdot \int_{x=0}^{x=1} \left(x - x^2\right) \cdot dx$$

(ऊपर − नीचे)

$(y = x)$ $(y = x^2)$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \cancel{2}\left(\frac{3-2}{\cancel{6}_3}\right) = \frac{1}{3}$$

P (Point of intersection)

$y = x$ — ①

$y = x^2$ — ②

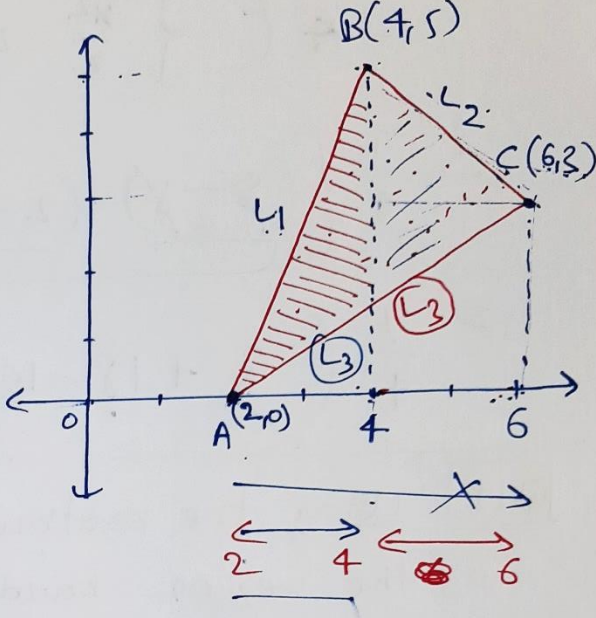
$\Rightarrow x = x^2$

$\Rightarrow x^2 - x = 0$

$x(x - 1) = 0$

$x = 0,\ x = 1$

$\boxed{\text{Q.13}}$ Using the method of integration find the area of triangle ABC, coordinates of whose vertices are A(2,0), B(4,5) and C(6,3).



Ans. $\boxed{(y-y_1) = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)}$

L_1 : AB

$$(y-0) = \left(\frac{5}{2}\right)(x-2)$$

$$\Rightarrow \boxed{y = \frac{5}{2}x - 5} : L_1$$

L_2 : BC $(y-3) = \left(\frac{\cancel{2}}{-\cancel{2}}\right)(x-6)$

$$\Rightarrow y = -x+6 \quad +3$$

$$\Rightarrow \boxed{y = -x+9} \longrightarrow L_2$$

L_3 : AC

$$(y-0) = \left(\frac{3}{4}\right)(x-2)$$

$$\Rightarrow \boxed{y = \frac{3x}{4} - \frac{3}{2}} \longrightarrow L_3$$

$$ar(\Delta ABC) = \int_2^4 (L_1 - L_3)\,.dx + \int_4^6 (L_2 - L_3)\,.dx$$

$$= \int_2^4 \left(\frac{5}{2}x - 5 - \frac{3x}{4} + \frac{3}{2}\right).dx + \int_4^6 \left(-x+9 - \frac{3x}{4} + \frac{3}{2}\right).dx$$

$$= \int_2^4 \left(\frac{7x}{4} - \frac{7}{2}\right).dx + \int_4^6 \left(-\frac{7x}{4} + \frac{21}{2}\right).dx$$

$$= \frac{7}{4}\int_2^4 (x-2).dx + \frac{7}{4}\int_4^6 (-x+6).dx$$

$$ar(\Delta ABC) = \frac{7}{4}\int_2^4 (x-2).dx + \frac{7}{4}\int_4^6 (-x+6).dx$$

$$= \frac{7}{4}\left[\left\{\frac{x^2}{2} - 2x\right\}_2^4 + \left\{-\frac{x^2}{2} + 6x\right\}_4^6\right]$$

$$= \frac{7}{4}\left[\underbrace{(\cancel{8}-\cancel{8}) - (2-4)} + \underbrace{(-18+36) - (-8+24)}\right]$$

$$= \frac{7}{4}(2+18-16) = \frac{7}{\cancel{4}}(\cancel{4}) = 7$$

$\boxed{Q.14}$ Using the method of integration find the area of the region bounded by lines: $2x+y=4 \rightarrow L_1$

$3x-2y=6 \rightarrow L_2$

$x-3y+5=0 \rightarrow L_3$

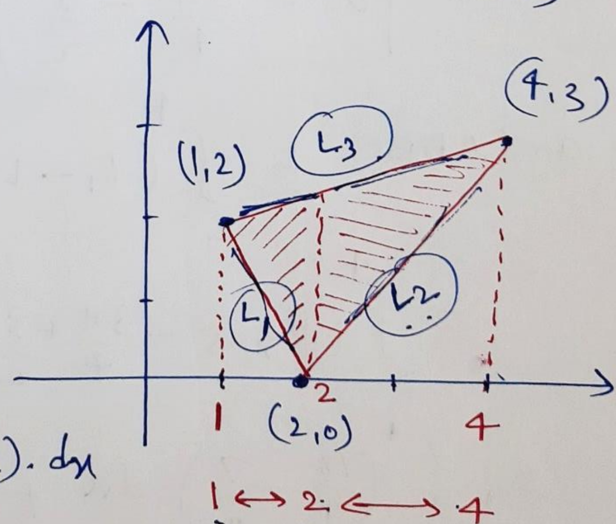
Ans.

$L_1\ L_2$	$L_2\ L_3$	$L_3\ L_1$	
$(2,0)$	$(4,3)$	$(1,2)$	← Point of intersection (vertices)

$L_1 \rightarrow 4-2x = y$

$L_2 \rightarrow y = \frac{3}{2}x - 3$

$L_3 \rightarrow y = \frac{x}{3} + \frac{5}{3}$



Required area = ar(Δ)

$$= \int_1^2 (L_3 - L_1).dx + \int_2^4 (L_3 - L_2).dx$$

$$= \int_1^2 \left(\frac{x}{3} + \frac{5}{3} - 4 + 2x\right).dx + \int_2^4 \left(\frac{x}{3} + \frac{5}{3} - \frac{3x}{2} + 3\right).dx$$

$$= \int_1^2 \left(\frac{7x}{3} - \frac{7}{3}\right) \cdot dx + \int_2^4 \left(-\frac{7x}{6} + \frac{14}{3}\right) \cdot dx$$

$$= \frac{7}{3}\left[\int_1^2 (x-1) \cdot dx + \int_2^4 \left(-\frac{x}{2} + \frac{2}{}\right) \cdot dx\right]$$

$$= \frac{7}{3}\left[\left(\frac{x^2}{2} - x\right)_1^2 + \left(-\frac{x^2}{4} + 2x\right)_2^4\right]$$

$$= \frac{7}{3}\left[\underbrace{(\cancel{2} - \cancel{2}) - \left(\frac{1}{2} - 1\right)} + \underbrace{(-4 + 8) - (-1 + 4)}\right]$$

$$= \frac{7}{3}\left[-\frac{1}{2} + 1 + 4 - 3\right] = \frac{7}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right) = \frac{7}{2}$$

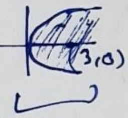
Q.15 Find the area of the region

$\{(x, y): y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}$

$x^2 + y^2 = \frac{9}{4}$

$r = \frac{3}{2}$ $(0, 0)$

Ans. $y^2 = 4x$

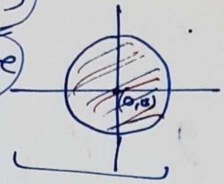


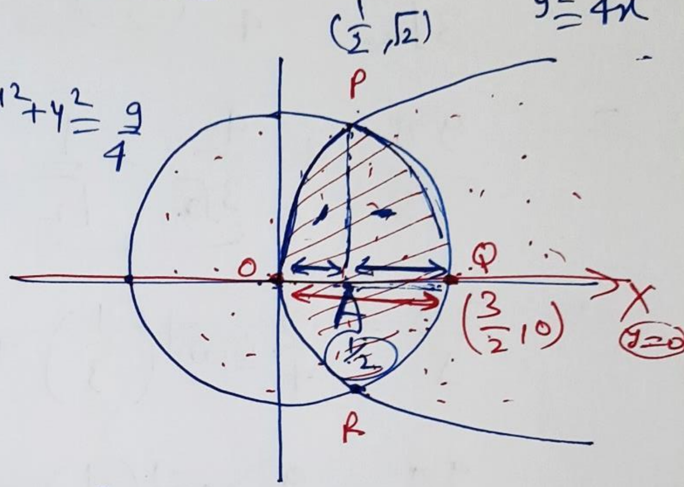
$0 \le 4 \times 3$

$0 \le 12$ True

$0 \le 9$

True





Required area

$= ar(OPQRO)$

$= 2 \times \boxed{ar(OPQO)}$

$= 2\left[\int_0^{1/2} (2\sqrt{x} - 0)\, dx + \int_{1/2}^{3/2} \left(\sqrt{\frac{9}{4} - x^2} - 0\right) dx\right]$

$P \rightarrow$ point of intersection

$y^2 = 4x$ $\quad 4x^2 + 4y^2 = 9$

$\Rightarrow x = \frac{1}{2}, \ \cancel{x = -\frac{9}{2}}$

$= 2\left[\int_0^{1/2} 2\sqrt{x}\, dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx\right]$

$= 2\left[2 \cdot \frac{2}{3} \left(x^{3/2}\right)_0^{1/2} + \left(\frac{x}{2}\sqrt{9/4 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right)_{1/2}^{3/2}\right]$

$= 2\left[\frac{4}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + \frac{3}{4}\cancel{\sqrt{\frac{9}{4} - \frac{9}{4}}} + \frac{9}{8}\sin^{-1}\left(\frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}\right) - \frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$

required area $= 2\left[\frac{\cancel{4}^{2}}{3}\left(\frac{1}{\cancel{2}\sqrt{2}}\right) + \frac{9}{8}\sin^{-1}(1) - \frac{1}{4}\sqrt{\frac{\cancel{8}2}{\cancel{4}}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$

$$= \frac{4}{3\sqrt{2}} + \frac{9}{4}\left(\frac{\pi}{2}\right) - \frac{1}{\cancel{2}}\frac{\cancel{\sqrt{2}}^{1}}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9\pi}{8} + \underbrace{\frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right).$$

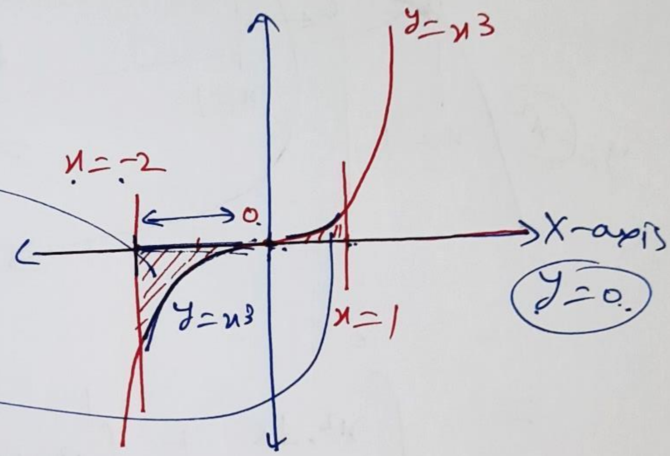
$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{4-3}{3\sqrt{2}}$$

$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}$$

Miscellaneous Exercise on chapter 8

Q.16 Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is —

(A) -9 (B) $\frac{-15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$.



Required area

$$= \int_{-2}^{0} (0 - x^3).dx + \int_{0}^{1} (x^3 - 0).dx$$

$$= \left(-\frac{x^4}{4}\right)_{-2}^{0} + \left(\frac{x^4}{4}\right)_{0}^{1} = [(0) - (-4)] + \left(\frac{1}{4} - 0\right)$$

$$= 4 + \frac{1}{4} = \frac{17}{4}$$

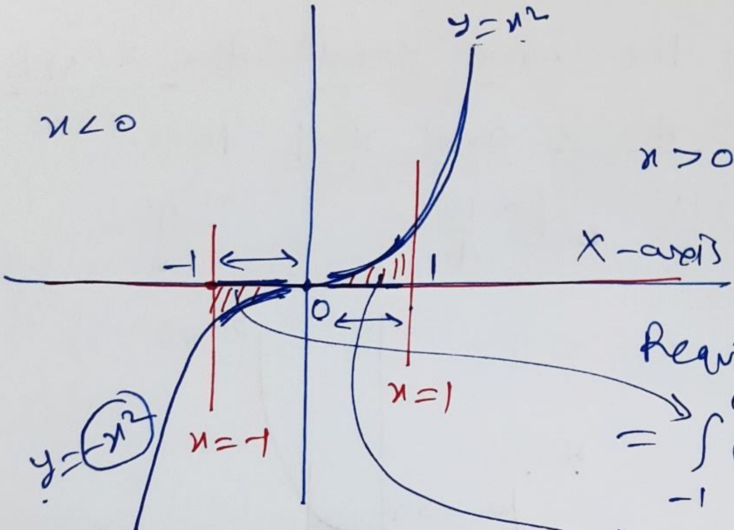
Q.17 The area bounded by the curve $\boxed{y = x|x|}$, x-axis and the ordinates $x = -1$ & $x = 1$ is given by —

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$y = x|x| = \begin{cases} x(x), & x > 0 \\ x(-x), & x < 0 \end{cases} = \begin{cases} x^2, & x > 0 \\ -x^2, & x < 0 \end{cases}$$

$y = x|x| = \begin{cases} x^2 , & x>0 \\ -x^2 , & x<0 \end{cases}$, $\boxed{\begin{array}{c} x\text{-axis} \\ x=-1 \\ x=1 \end{array}}$



$x>0$

X-axis ($y=0$)

Required area

$$= \int_{-1}^{0} (0-(-x^2)) \cdot dx + \int_{0}^{1} (x^2-0) \cdot dx$$

$$= \int_{-1}^{0} x^2 \cdot dx + \int_{0}^{1} x^2 \cdot dx = \left(\frac{x^3}{3}\right)_{-1}^{0} + \left(\frac{x^3}{3}\right)_{0}^{1}$$

$$= \left(0+\frac{1}{3}\right) + \left(\frac{1}{3}-0\right) = \frac{2}{3}$$

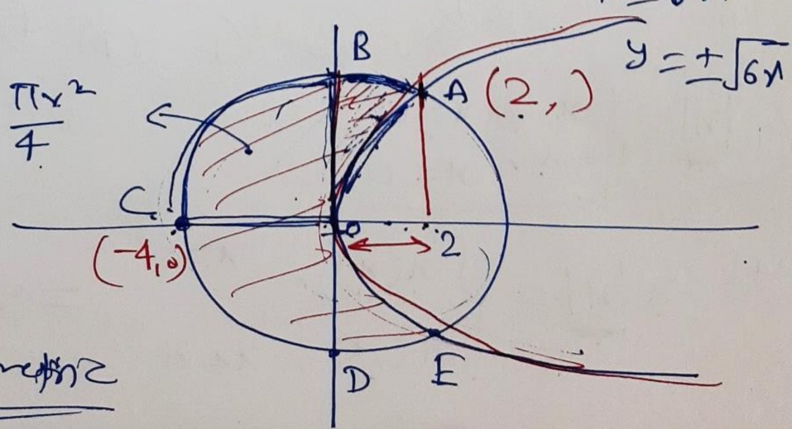
Q.18 The area of the circle $\boxed{x^2+y^2=16}$, exterior to the parabola $y^2=6x$ is ——

$x^2+y^2=16=4^2$

$(0,0)$, $r=4$

$4\pi = \frac{\pi r^2}{4}$

Ex: interior



Symmetric

required area $= ar(OABCDEO) = 2 \cdot ar(OABCO)$

$= 2\left(ar(OBCO) + ar(OABO) \right)$

Quadrant 

$\frac{\pi r^2}{4} = \frac{\pi 4^2}{4} = 4\pi$

$\int_0^2 \left(\sqrt{16-x^2} - \sqrt{6x}\right) \cdot dx$ (Circle − Parabola)

$$\left(\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right) - \sqrt{6}\,\frac{2}{3}x^{3/2} \right)_0^2$$

$$= \left(2\sqrt{3} + 8\sin^{-1}\left(\frac{1}{2}\right) - \sqrt{6}\cdot\frac{2}{3}\cdot 2\sqrt{2}\right)$$

$$= \left(2\sqrt{3} - \frac{\sqrt{2}\cdot 3\cdot 4\sqrt{2}}{3} + 8\cdot\frac{\pi}{6}\right)$$

$$= \left(-\frac{2\sqrt{3}}{3} + \frac{4\pi}{3}\right)$$

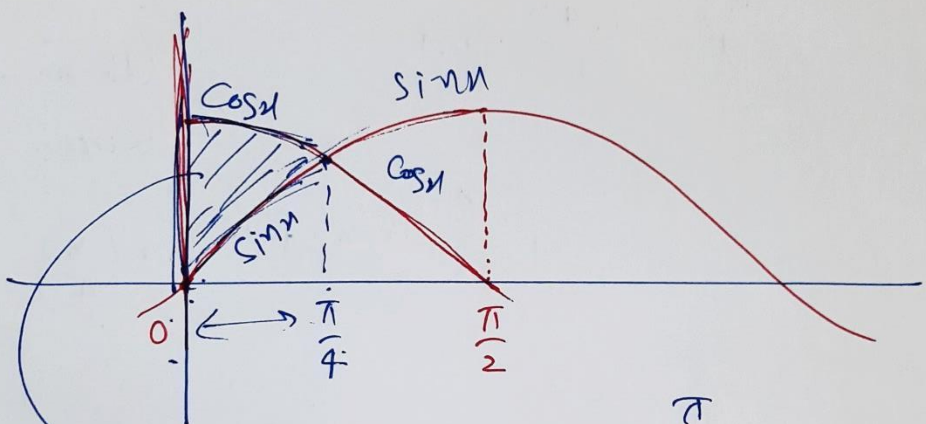
$$= 2\left(4\pi - \frac{2\sqrt{3}}{3} + \frac{4\pi}{3}\right)$$

$$= 2\left(\frac{12\pi - 2\sqrt{3} + 4\pi}{3}\right) = \frac{2}{3}\left(16\pi - 2\sqrt{3}\right)$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right)$$

$\boxed{Q.19}$ The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$, when $0 \le x \le \frac{\pi}{2}$ is—

(A) $2(\sqrt{2}-1)$ (B) $\sqrt{2}-1$ (C) $\sqrt{2}+1$ (D) $\sqrt{2}$



Required area $= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$

$$= \left(\sin x + \cos x\right)_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)$$

$2 = \sqrt{2} \times \sqrt{2}$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$