DETERMINANTS [सारणिक]

Determinant is a number associated with a square matrix.

Determinant.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $\qquad |A| = \Delta = \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$= -2$?

Determinant of 'B'

$B = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}_{3\times 3}$ $\qquad |B| = \Delta = \det(B) = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$= -28$?

input $\longrightarrow$ $\boxed{}$ $\longrightarrow$ output

Machine

$f(x) = x^2 + 2x + 1$

Square matrix $\longrightarrow$ $\boxed{}$ $\longrightarrow$ Determinant

Function

Number

only on a number

Modulus

$|-3| = 3$

$|-5| = 5$

Determinant of 1×1 matrix.

$A = [a_{ij}]_{1\times 1} = [a_{11}]_{1\times 1}$

$\det(A) = |A| = \Delta = |a_{11}| = a_{11}$

Direct (as it is)

e.g.

$A = [2] \longrightarrow |A| = 2$

$B = [-5] \Longrightarrow |B| = \det(B) = |-5| = -5$

Determinant of 2X2 matrix

$$A = [a_{ij}]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

row ↑ 2 × 2 ↑ Column

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (\searrow) - (\nearrow)$$

$$= a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

e.g.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6$$

$$= -2$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \Rightarrow |B| = 4 - 6 = -2$$

Determinant of 3X3 matrix.

$$A = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

rows = 3
Columns = 3
total = 6
↓
ways.
तरीके

$$|A| = \det(A) = \Delta = \begin{array}{c|ccc|} R_1 \rightarrow & a_{11} & a_{12} & a_{13} \\ R_2 \rightarrow & a_{21} & a_{22} & a_{23} \\ R_3 \rightarrow & a_{31} & a_{32} & a_{33} \\ & \uparrow & \uparrow & \uparrow \\ & C_1 & C_2 & C_3 \end{array}$$

By expanding along R_1 :→

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$(-1)^{1+1} = (-1)^2 = +1$

$(-1)^{1+2} = (-1)^3 = -1$ $\quad (-1)^{1+3} = (-1)^4 = 1$

even → (+)
odd → (−)

$(-1)^{i+j} (a_{ij})$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31} \cdot a_{23})$$
$$+ a_{13}(a_{21} \cdot a_{32} - a_{31}a_{22})$$

R_1 ✓ $\quad R_2$ ✓ $\quad R_3$ ✓ $\quad C_1$ ✓ $\quad C_2$ ✓ $\quad C_3$ ✓

Note: Always try to expand along that row or column which has more zero.

e.g. $|B| = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ $\quad$ By expanding along C_2

$$|B| = -(-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 0 \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} - 5 \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$$

$(-1)^{1+2}$ $\quad$ $(-1)^{3+2}$

2×2 $\quad$ 0

$= 3(-42 - 4) + 0 - 5(8 - 30)$

$= 3 \times (-46) - 5(-22) = -138 + 110 = -28$

SARRUS METHOD (to solve 3x3 Determinant)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= (a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13} \cdot a_{21} \cdot a_{32})$$
$$- (a_{31} \cdot a_{22} \cdot a_{13} + a_{32}\, a_{23}\, a_{11} + a_{33}\, a_{21}\, a_{12})$$

e.g. By Sarrus method.

$$|B| = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}_{3\times 3} \begin{matrix} 2 & -3 \\ 6 & 0 \\ 1 & 5 \end{matrix}$$

$$= (0 - 12 + 150) - (0 + 40 + 126)$$
$$= 138 - 166$$
$$= -28$$

Note :

(I) $A = 2B$, A: 2×2 matrix, order $= 2$; B: 2×2, order $= 2$

Determinant

$|A| = |2B|$

$|A| = 2^2 |B|$

$|A| = 4|B|$

(II) $A = 2B$, A: 3×3, order $= 3$; B: 3×3, order $= 3$

$|A| = |2B|$

$|A| = 2^3 |B|$

$|A| = 8|B|$

e.g.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}_{2\times2}$

$$\boxed{|A| = \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = 0 - 6 = -6}$$

$-6 \times 4 = -24$, $4 = 2^2$

$B = 2A = 2\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}_{2\times2}$

$B = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}$

$|B| = \left|\begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}\right|$

$= 0 - 24$

$= -24$

Generalise

$A = KB$ $\quad$ $A, B \rightarrow n\times n$

$|A| = |KB|$ $\quad$ $K =$ constant

$|A| = K^n |B|$

Determinants   Exercise (4.1)

(3×3)

(Q.1) $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}_{2\times 2} = \underbrace{2(-1)} - \underbrace{(-5)\,4}$

$= -2 + 20 = 18$

Q2 (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta)$

$= \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x+1)(x^2-x+1)$

$- (x+1)(x-1)$

$= x^3 - \cancel{x^2} + \cancel{x} + \cancel{x^2} - \cancel{x} + 1 - x^2 + 1$

$= x^3 - x^2 + 2$

Q.3 $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ To Prove,

$|2A| = 4|A|$

LHS $= |2A|$

$= \left| 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \right|$

$= \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$

$= 8 - 32$

$= -24$

RHS $= 4|A|$

$= 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$

$= 4(2-8)$

$= 4(-6)$

$= -24$

Q.4

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$$

To Prove

$$|3A| = 27|A|$$

$$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

LHS $= |3A|$

$$= \begin{array}{c} \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}_{3\times 3} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{array} \end{array} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \\ \\ \\ \end{array}$$

By expanding along 'C_1'

$$= +3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} \to 0$$

$$+ 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} \to 0$$

$$= 3(36-0) = 108$$

RHS $= 27\,|A|$

$$= 27 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} \to R_3$$

along R_3

$$= 27 \cdot \left\{ +0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right\}$$

$$= 27 \cdot \{4(1-0)\}$$

$$= 108$$

$$\boxed{\text{LHS} = \text{RHS}}$$

$\boxed{Q.5}$

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

I-method (Boards)
expand along any row / column

II-method (Sarrus method)
↓
3×3

Short cut

II-method, Sarrus method

$$\Delta = \left|\begin{array}{ccc|cc} 3 & -1 & -2 & 3 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 3 & -5 & 0 & 3 & -5 \end{array}\right.$$

$(\searrow + \searrow + \searrow) - (\nearrow + \nearrow + \nearrow)$

$= (0 + 3 + 0) - (0 + 15 + 0)$

3×0×0

$= 3 - 15 = -12$

(ii) $$\left|\begin{array}{ccc|cc} 3 & -4 & 5 & 3 & -4 \\ 1 & 1 & -2 & 1 & 1 \\ 2 & 3 & 1 & 2 & 3 \end{array}\right.$$

(Sarrus method)

$= (3 + 16 + 15) - (10 - 18 - 4)$

$= 34 + 12$

$= 46$

Q.5 (III) (IV)

&

Q6

Same approach

$\boxed{Q.7}$ Find 'x'

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

(2×2) (2×2)

Similarly

$\Rightarrow (2-20) = 2x^2 - 24$

$\Rightarrow -18 = 2x^2 - 24$

$\Rightarrow 24 - 18 = 2x^2$

$\Rightarrow 2x^2 = 6$

$\Rightarrow 2x^2 = 6$

$x^2 = 3$

$\boxed{x = \pm\sqrt{3}}$

$\boxed{Q.8} \rightarrow$ Similarly.

Properties of Determinants (सारणिक के गुणधर्म)

Prop. ① Matrix A ← square matrix

$|A| = |A^T|$

e.g. $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix}$

$\Rightarrow (-28) = (-28)$

Prop. ②: If two rows (Columns) are interchanged $R_i \leftrightarrow R_j$

then sign of determinant also changes.

e.g. $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = -28$

R_2

R_3

$(R_2 \leftrightarrow R_3)$

$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix} = 28$

Prop. ③: if two rows (Columns) are identical

then $\Delta = 0$

e.g. $\begin{vmatrix} \boxed{2 \quad 3 \quad 5} \\ 7 \quad -6 \quad 0 \\ \boxed{2 \quad 3 \quad 5} \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 2 & 2 \\ -7 & 5 & 5 \\ 0 & -8 & -8 \end{vmatrix} = 0$

Prop. 4 If determinant is multiplied by 'K' then only one row (or one column) is multiplied by 'K'.

Matrix		Determinant
$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$	v/s	$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}_{3\times 3}$
$2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$		$2\Delta = 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$
$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$		$\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ $R_1 \uparrow 2$ $C_3 \uparrow 2$ $\begin{vmatrix} 1 & 2 & 6 \\ 4 & 5 & 12 \\ 7 & 8 & 18 \end{vmatrix}$

$$\underset{3\times 3}{A} = K \underset{3\times 3}{B}$$

$$|A| = |KB|$$

$$|A| = K^3 |B|$$

Results: (I) If we take common 'K' from only one row, then determinant can be written as

$$\Delta = \begin{vmatrix} Ka & Kb & Kc \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \Delta = K \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Result ② If elements of two (rows) (or columns) are proportional then $\Delta = 0$.

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & -6 & 0 \\ 6 & 9 & 15 \end{vmatrix} = 0 = 3\begin{vmatrix} 2 & 3 & 5 \\ 7 & -6 & 0 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

$2 \times 3 \quad 3 \times 3 \quad 5 \times 3$

Property ⑤

$$\begin{vmatrix} a+x & b+y & c+z \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & y & z \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 11 & 4 \end{vmatrix}$$

v/s

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 11 & 8 \end{bmatrix}$$

Property ⑥ Elementary operation

$R_i \to R_i \pm K R_j$ या $C_i \to C_i \pm K C_j$

☆ (changing row should not be used in changing other row) Simultaneously

$R_1 \to R_1 + 2R_2$

~~$R_3 \to R_3 - 3R_1$~~

e.g.

$$\Delta = \begin{vmatrix} 101 & 102 & 103 \\ 104 & 105 & 106 \\ 107 & 108 & 109 \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \end{matrix}$$

Property

$\underbrace{R_2 \to R_2 - R_1}$, $\underbrace{R_3 \to R_3 - R_1}$ ($R_1 \to$ no change)

$$\Delta = \begin{vmatrix} 101 & 102 & 103 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

$104 - 101 = 3$

$107 - 101 = 6$

R_3 के Elements & R_2 के elements

Proportional

$\Rightarrow \boxed{\Delta = 0}$

Note! If all elements of one row (or one column.) are zero then $\Delta = 0$

e.g.

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 3 & 18 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0$$

e.g.

$$\begin{vmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{vmatrix} = 0$$

e.g. To Prove: $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = (1+xyz).(x-y)(y-z)(z-x)$

$$LHS = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

By Prop. (5)

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \begin{matrix} \to x \\ \to y \\ \to z \end{matrix}$$

$\boxed{C_1 \leftrightarrow C_2}$ R₁ में से 'x' common

$$= -\underbrace{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}} + xyz \underbrace{\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}}$$

$C_1 \leftrightarrow C_3$

$$= + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1+xyz). \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\Delta = (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \begin{array}{l} \text{Common} \\ \to (y-x) \\ \to (z-x) \end{array}$$

$$= (1+xyz)(y-x)(z-x).\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

By expanding along 'C_1'

$$= (1+xyz)(y-x)(z-x).\left\{1.\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} - 0|\;| + 0|\;|\right\}$$

$$= (1+xyz)(y-x)(z-x).\{z+\cancel{x}-y-\cancel{x}\}$$

$$= (1+xyz)(y-x)(z-x)(z-y)$$

$$\ominus \qquad \qquad \ominus$$

$$= +(1+xyz)(x-y)(y-z)(z-x) = RHS.$$

e.g. To Prove,

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \underbrace{abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} = \underbrace{abc+ab+bc+ca}$$

$$LHS = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

take common 'a' from R_1
b from R_2
c from R_3

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} \; abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

C_3

By expanding along C_3

$$\Delta = (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot\left\{+1\begin{vmatrix} 1 & 0 \\ \frac{1}{b} & 1 \end{vmatrix}\right\}$$

$$= (abc)\cdot\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot 1.(1-0)$$

$$= abc\cdot\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= abc+bc+ac+ab$$

H.P.

Exercise (4.2) Determinants (Properties)

Properties of Determinants.

$\Delta = \det(A)$

1. $|A^T| = |A|$
2. $R_i \leftrightarrow R_j \rightsquigarrow \Delta \longrightarrow -\Delta$
3. 2 rows identical $\Longrightarrow \Delta = 0$

$\boxed{\text{all elements of any row } = 0 \Rightarrow \Delta = 0}$

4. $k\Delta \Leftrightarrow k$ (1 row)
   या
   k (1 column)

$\boxed{\text{2 rows proportional} \Rightarrow \Delta = 0}$

5. $\begin{vmatrix} \cdot\cdot & \cdot\cdot & \cdot\cdot \\ - & - & - \\ - & - & - \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & \cdot \\ - & - & - \\ - & - & - \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot \\ - & - & - \\ - & - & - \end{vmatrix}$

6. $R_i \longrightarrow R_i \pm kR_j$

Exercise 4.2 (Prove without expanding)

$\boxed{Q.1}$

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

$$LHS = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$$

Property no. 5

$$= \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0 + 0 = 0 = RHS$$

C_1, C_3 identical

C_2 C_3 identical

Q-2

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} a-b & b-c & c-a \\ \boxed{b-c} & \boxed{c-a} & \boxed{a-b} \\ \boxed{c-a} & a-b & b-c \end{vmatrix}$$

$$\cancel{a}-\cancel{b}+\cancel{b}-\cancel{c}+\cancel{c}-\cancel{a} = 0$$

Operation $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 = \text{RHS}.$$

all elements are zero

Q.3

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

C_1 C_2

$9 \times C_2 + C_1 = C_3$

$7 \times 9 + 2 = 65$

$8 \times 9 + 3 = 75$

$9 \times 9 + 5 = 86$

81

$C_3 \to C_3 - C_1 - 9C_2$

$65 - 2 - \underbrace{9 \times 7}$

$\cancel{65} - \cancel{65} = 0$

$$= \begin{vmatrix} 2 & 7 & \boxed{0} \\ 3 & 8 & \boxed{0} \\ 5 & 9 & \boxed{0} \end{vmatrix} = 0 = \text{RHS}.$$

$\boxed{Q.4}$

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

$$LHS = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{array}{c} \begin{vmatrix} 1 & \underline{bc} & \underline{ab+ac} \\ 1 & \underline{ca} & \underline{bc+ba} \\ 1 & \underline{ab} & \underline{ca+cb} \end{vmatrix} \\ \qquad\quad C_2 \qquad C_3 \end{array}$$

$ab + \underline{bc} + ca$

$$\boxed{C_3 \to C_3 + C_2}$$

$$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

C_3 में से 'ab+bc+ca' Common

$$= (ab+bc+ca) \begin{array}{c} \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \\ C_1 \qquad\quad C_3 \end{array} = 0 = RHS.$$

C_1 & C_3 के elements = identical

Exercise 4.2 (Properties of determinants)

Q.5 $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

$2a+2b+2c$

LHS = $\begin{vmatrix} \boxed{b+c} & q+r & y+z \\ \boxed{c+a} & r+p & z+x \\ \boxed{a+b} & p+q & x+y \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} \underline{2a+2b+2c} & \underline{2p+2q+2r} & \underline{2x+2y+2z} \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

$\underline{R_2 \to R_2 - R_1}$ & $\underline{R_3 \to R_3 - R_1}$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} \begin{matrix} \\ \to (-1) \\ \times \\ \to (-1) \\ = (+1) \end{matrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ b & q & y \\ c & r & z \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix}$$

$R_1 \rightarrow R_1 - R_2 - R_3$ $\quad (a+\cancel{b}+\cancel{c}) - (\cancel{b}) - (\cancel{c})$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = RHS.$$

$\boxed{Q.6}$

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0 \qquad |A| = 0$$

matrix

$$A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

(Skew symmetrix)

$\boxed{A^T = -A}$

$$A^T = \begin{bmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{bmatrix} = -A$$

$$LHS = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}_{3\times 3} = |A|$$

Property

$|A^T| = |A|$

$\because$ 'A' is skew sym. $\quad A^T = -A$

$\Rightarrow |-A| = |A|$

$\Rightarrow |(-1).A| = |A|$

$\boxed{|A| = |kB| \quad (A, B \rightarrow n\times n) \\ |A| = k^n\,|B|}$

$\Rightarrow (-1)^3\,|A| = |A|$ $\qquad (-1)^3 = -1$

$\Rightarrow -|A| = |A|$

$\Rightarrow 0 = 2|A|$ $\qquad \Rightarrow 0 = |A|$

Hence Proved.

$\boxed{Q.7}$

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

$$LHS = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} \begin{array}{l} \rightarrow R_1 \text{ में से 'a' common} \\ \rightarrow R_2 \rightarrow b \\ \rightarrow R_3 \rightarrow c \end{array}$$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \\ (a) & (b) & (c) \end{array} \text{ common.}$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \end{array}$$

$R_1 \rightarrow R_1 + R_2$

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \rightarrow R_1 \text{ के along expand}$$

$$= a^2b^2c^2 \cdot \left\{ \cancel{0}\,|\ \ |^{\,0} - \cancel{0}\,|\ \ |^{\,0} + 2\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \right\}$$

$$\Rightarrow a^2b^2c^2 \cdot \{ 2(1+1) \} = 4a^2b^2c^2$$

$$= RHS.$$

Exercise-4.2 (Properties of Determinants)

Q.8 (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(i) LHS $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$

$\boxed{R_2 \rightarrow R_2 - R_1}$ & $\boxed{R_3 \rightarrow R_3 - R_1}$

$= \begin{vmatrix} 1 & a & a^2 \\ \boxed{0 & b-a & b^2-a^2} \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$\rightarrow b^2 - a^2 = (b-a)(b+a)$

$\rightarrow R_2$ में से $(b-a)$ common

$\rightarrow R_3 \rightarrow (c-a)$ common.

$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

by expanding along (C_1)

$= (b-a)(c-a). \left\{ 1 \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} - \underbrace{0|\,\,|}_{0} + \underbrace{0|\,\,|}_{0} \right\}$

$= (b-a)(c-a).(c + \cancel{a} - b - \cancel{a})$

$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a) = $ RHS.

(ii)

$$\text{LHS} = \begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{matrix} \end{array}$$

$C_2 \rightarrow C_2 - C_1$

&

$C_3 \rightarrow C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

$b^3+a^3 = (b+a)(b^2-ab+a^2)$

☆ $b^3-a^3 = (b-a)(b^2+ab+a^2)$

common $C_2 \rightarrow (b-a)$ common $C_3 \rightarrow (c-a)$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix}$$

$$\begin{array}{r} c^2+ac+\cancel{a^2} \\ -b^2-ab-\cancel{a^2} \\ \hline = c^2-b^2 + a(c-b) \end{array}$$

$\rightarrow (c-b)(c+b) + a(c-b)$

$\boxed{C_3 \rightarrow C_3 - C_2}$ apply.

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^3 & b^2+ab+a^2 & (c-b)(c+b)+a(c-b) \end{vmatrix}$$

Common $(c-b)$

$$= (b-a)(c-a)(c-b) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^3 & b^2+ab+a^2 & c+b+a \end{vmatrix}$$

R_1 along expand

$$= (a-b)(c-a)(b-c) \cdot \left\{ 1 \begin{vmatrix} 1 & 0 \\ b^2+ab+a^2 & c+b+a \end{vmatrix} - 0 \, | \, | + 0 \, | \, | \right\}$$

$$= (a-b)(b-c)(c-a) \cdot \{ a+b+c - 0 \}$$

$$= (a-b)(b-c)(c-a) \cdot (a+b+c) = RHS.$$

$\boxed{Q.9}$ $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x) \cdot (xy+yz+zx)$

$$LHS = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

$\boxed{R_2 \to R_2 - R_1}$ & $\boxed{R_3 \to R_3 - R_1}$

$$= \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix} = \begin{vmatrix} x & x^2 & yz \\ (y-x) & (y-x)(y+x) & -z(y-x) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix}$$

$R_2 \to (y-x)$ Common

$R_3 \to (z-x)$ common

$$= (y-x)(z-x) \begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 1 & z+x & -y \end{vmatrix} \begin{array}{l} \\ \to R_2 \\ \to R_3 \end{array}$$

$\boxed{R_3 \to R_3 - R_2}$

$$= (y-x)(z-x) \begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 0 & z-y & z-y \end{vmatrix} \begin{array}{l} \\ \\ \to R_3 \ (z-y) \end{array}$$

Common

$$= (y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 0 & 1 & 1 \end{vmatrix}$$

$$\begin{array}{ccc} & C_2 & C_3 \end{array}$$

$$\boxed{C_2 \to C_2 - C_3}$$

$$= (y-x)\,(z-x)(z-y)\begin{vmatrix} x & x^2-yz & yz \\ 1 & x+y+z & -z \\ 0 & 0 & 1 \end{vmatrix} \to R_3 \text{ के}$$

$\ominus \quad ✓ \quad \ominus$

along expansion

$$= (x-y)(y-z)(z-x)\cdot\left\{0\,|\;\;| - 0\,|\;\;| + 1\begin{vmatrix} x & x^2-yz \\ 1 & x+y+z \end{vmatrix}\right\}$$

$$= (x-y)(y-z)(z-x)\cdot\left\{\cancel{x^2} + xy + xz - \cancel{x^2} + yz\right\}$$

$$= (x-y)(y-z)(z-x)\cdot\left\{xy + yz + zx\right\}$$

$= RHS$

Exercise (4.2) {Properties of Determinants}

Q. 10

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

LHS $= \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{matrix}$

$(x+4) + (2x) + (2x) = 5x+4$

$C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix} = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$

C_1 में $(5x+4)$ common

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4-x \end{vmatrix}$

C_1 के along expansion

$= (5x+4) \cdot \left\{ 1 \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix} - 0 + 0 \right\}$

$= (5x+4) \cdot (4-x)^2 = \text{RHS}$

(ii) $\begin{vmatrix} y+K & y & y \\ y & y+K & y \\ y & y & y+K \end{vmatrix} = K^2(3y+K)$

$$\text{LHS} = \begin{vmatrix} y+K & y & y \\ y & y+K & y \\ y & y & y+K \end{vmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{array}$$

$y+K + y + y = 3y+K$

$\boxed{C_1 \to C_1 + C_2 + C_3}$

$$= \begin{vmatrix} 3y+K & y & y \\ 3y+K & y+K & y \\ 3y+K & y & y+K \end{vmatrix} = (3y+K)\begin{vmatrix} 1 & y & y \\ 1 & y+K & y \\ 1 & y & y+K \end{vmatrix}$$

C_1 में $(3y+K)$ common

$\boxed{\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}}$

$$= (3y+K)\begin{vmatrix} 1 & y & y \\ 0 & K & 0 \\ 0 & 0 & K \end{vmatrix}$$

C_1 के along expansion

$$\Rightarrow (3y+K) \cdot \left\{ 1 \begin{vmatrix} K & 0 \\ 0 & K \end{vmatrix} - \cancel{0 \,|\;|} + \cancel{0\,|\;|} \right\}$$

$$= (3y+K) \cdot (K^2) = \text{RHS}$$

Q.11 (i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

$$LHS = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & C_2 & C_3 \end{array}$$

$$\boxed{\begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array}}$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix} \to R_1 \text{ के along expand}$$

$$= (a+b+c) \cdot \left\{ 1 \begin{vmatrix} -a-b-c & 0 \\ 0 & -a-b-c \end{vmatrix} - \cancel{0 \left| \quad \right|} + \cancel{0 \left| \quad \right|} \right\}$$

$$= (a+b+c) \cdot (a+b+c)^2 = (a+b+c)^3 = RHS.$$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

$$LHS = \begin{array}{c} \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} \\ \begin{matrix} \uparrow & \quad\uparrow & \quad\uparrow \\ C_1 & \quad C_2 & \quad C_3 \end{matrix} \end{array}$$

$\boxed{C_1 \to C_1 + C_2 + C_3}$

$$= \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ \underbrace{2x+2y+2z} & x & z+x+2y \end{vmatrix}$$

C_1 में $\underline{2(x+y+z)}$ common

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & x+z+2y \end{vmatrix}$$

$\boxed{\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}}$

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

C_1 के along expansion.

$$= 2(x+y+z) \cdot \left\{ 1 \cdot (x+y+z)^2 - 0(\ ) + 0(\ ) \right\} = RHS$$

Q.12

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

$$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

$$1-x^3 = 1^3-x^3$$

$$=(1-x)^1(1+x+x^2)^1$$

$$\text{LHS} = \begin{array}{c} \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ C_1 \quad C_2 \quad C_3 \end{array}$$

$$\boxed{C_1 \to C_1+C_2+C_3}$$

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

C_1 में $(1+x+x^2)$ common

$$\boxed{\begin{array}{l} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{array}}$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

$x-x^2 = x(1-x)$

$x^2-x = -x(1-x)$

$1-x^2 = (1-x)(1+x)$

(R_2 में से $(1-x)$
R_3 में से $(1-x)$ > common)

$$= (1-x)^2(1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

C_1 के along expansion,

$$= (1-x)^2(1+x+x^2) \cdot \{1(1+x+x^2)\}$$

$$= [(1-x)\cdot(1+x+x^2)]^2 = [1-x^3]^2 = \text{RHS}.$$

Exercise (4.2) $\boxed{\text{Properties of Determinants}}$

$\boxed{\text{Q.13}}$ $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

$$\text{LHS} = \begin{array}{c} \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \\ \begin{matrix} \uparrow \quad\quad & \uparrow \quad\quad & \uparrow \\ C_1 \quad\quad & C_2 \quad\quad & C_3 \end{matrix} \end{array}$$

$\boxed{C_1 \to C_1 - bC_3}$ $\boxed{C_2 \to C_2 + aC_3}$

$$= \begin{vmatrix} (1+a^2+b^2) & 0 & -2b \\ 0 & (1+a^2+b^2) & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

C_1 & C_2 में से $(1+a^2+b^2)$ common

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

By applying $R_3 \to (R_3 - bR_1 + aR_2)$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix} \longrightarrow R_3 \text{ के along expand}$$

$$= (1+a^2+b^2)^2 \cdot \left\{ \cancel{+0} \left| \quad \right|^{0} - \cancel{0} \left| \quad \right|^{0} + (1+a^2+b^2) \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{(1-0)=1} \right\}$$

$$= (1+a^2+b^2)^2 \cdot (1+a^2+b^2)$$

$$= (1+a^2+b^2)^3 = RHS.$$

$\boxed{Q.14}$

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = \underbrace{1+a^2+b^2+c^2}$$

$$LHS = \begin{vmatrix} \underline{a^2+1} & ab & ac \\ ab & \underline{b^2+1} & bc \\ ca & cb & \underline{c^2+1} \end{vmatrix} \begin{matrix} \longrightarrow R_1 \longrightarrow \text{'a' common} \\ \longrightarrow R_2 \longrightarrow \text{'b' common} \\ \longrightarrow R_3 \longrightarrow \text{'c' common} \end{matrix}$$

$$= abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

$\boxed{C_1 \text{ पर}}$ $\boxed{C_2 \text{ पर}}$ $\boxed{C_3 \text{ पर}}$

C_1 $\quad C_2$ $\quad C_3$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \quad \boxed{C_1 \to C_1+C_2+C_3}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & 1+b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & 1+c^2 \end{vmatrix}$$

C_1 में से $(1+a^2+b^2+c^2)$ common

$$= (1+a^2+b^2+c^2) \begin{array}{c} \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & 1+b^2 & c^2 \\ 1 & b^2 & 1+c^2 \end{vmatrix} \end{array} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

$$\boxed{\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

C_1 के along expand.

$$= (1+a^2+b^2+c^2) . \left\{ 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \; + 0 \right\}$$

$(1-0)$ $\quad 0 \quad 0$

$$= (1+a^2+b^2+c^2) = \text{RHS}.$$

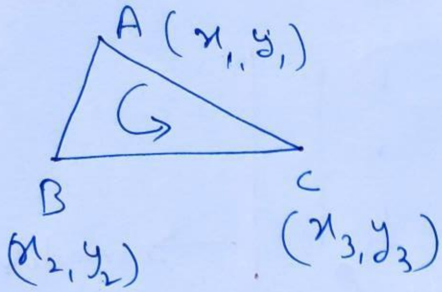
$\boxed{\text{Q. 15}}$ $|KA| \to n \times n$

$K^n |A|$ $\quad 3\times 3$

$K^3 |A|$

Determinants

Area of a Triangle



10th class

$$\text{Area} = \left| \frac{1}{2} \left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right] \right|$$

$\boxed{\text{Area} = \text{+ve}}$

12th Class 'Determinant'

$$\text{Area} = \left| \frac{1}{2} \underbrace{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}_{} \right|$$

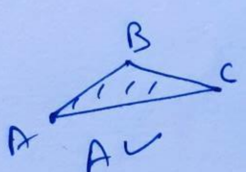
3X3

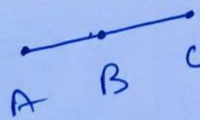
→ modulus

Note. ① If area is given, then $\frac{1}{2}\Delta = \pm A$

A

② If 3 points are collinear (संरेखीय), then area(ABC) = 0
(A, B, C)





Area = 0

e.g. Find area of $\triangle ABC$, $A(3,8)\ B(5,1)\ C(-4,2)$.

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ 5 & 1 & 1 \\ -4 & 2 & 1 \end{vmatrix} \right| \longrightarrow \text{modulus}$$

Determinant

$=$ By expanding along R_1

$$\left| \frac{1}{2} \left\{ 3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 8 \begin{vmatrix} 5 & 1 \\ -4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 5 & 1 \\ -4 & 2 \end{vmatrix} \right\} \right|$$

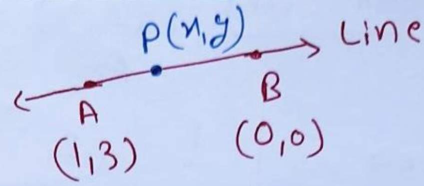
$$= \left| \frac{1}{2} \left\{ 3(1-2) - 8(5+4) + 1(10+4) \right\} \right|$$

$$= \left| \frac{1}{2} \left\{ -3 - 72 + 14 \right\} \right|$$

$$= \left| -\frac{61}{2} \right| = \frac{61}{2} \text{ sq. units}$$

e.g. Find the equation of the line joining $A(1,3)$ & $B(0,0)$ using determinants and find K if $D(K,0)$ is a point such that area of ΔABD is 3 square units.

Ans. Let $P(x,y)$ lies on line joining AB.



$\therefore$ P, A. & B are collinear.

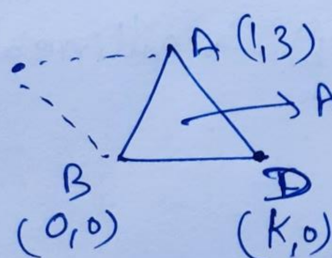
$ar(\Delta PAB) = 0 \Rightarrow \frac{1}{2}\left|\begin{vmatrix} 1 & 3 & 1 \\ x & y & 1 \\ 0 & 0 & 1 \end{vmatrix}\right| = 0$

$\Rightarrow \left|\begin{array}{ccc|cc} 1 & 3 & 1 & 1 & 3 \\ x & y & 1 & x & y \\ 0 & 0 & 1 & 0 & 0 \end{array}\right| = 0$ (Sarrus method)

$\Rightarrow (y + 0 + 0) - (0 + 0 + 3x) = 0$

$\Rightarrow \boxed{y - 3x = 0}$ Line.

next Part



Area $= 3 = \left|\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ K & 0 & 1 \end{vmatrix}\right|$ ± 3

$\Rightarrow \frac{1}{2}\left\{-\cancel{3}\begin{vmatrix} 0 & 1 \\ K & 1 \end{vmatrix}\right\} = \pm\cancel{3}$ ± 1

$\Rightarrow -(0 - K) = \pm 2$

Exercise 4.3

Area of a Triangle using Determinants

Q.1 Part (ii) $(2,7)$ $(1,1)$ $(10,8)$

$A \quad B \quad C$

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} \right|$$

R_1 के along expand

$$= \left| \frac{1}{2} \left\{ 2 \begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} - 7 \begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix} \right\} \right|$$

$$(1-8) \qquad (1-10) \qquad (8-10)$$

$$= \left| \frac{1}{2} \left\{ -14 + 63 - 2 \right\} \right|$$

$$= \left| \frac{1}{2} \{47\} \right| = \frac{47}{2} \text{ sq. units}$$

Q.2 $A(a, b+c)$ $B(b, c+a)$ $C(c, a+b)$ → collinear

$ar(\Delta ABC) = 0$ To Prove.

$$\underbrace{\frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}}_{LHS} = \underbrace{0}_{RHS} \leftarrow \text{to prove}$$

$$LHS = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

C_1

2020-21

$$= \frac{1}{2}\left\{ a\begin{vmatrix} c+a & 1 \\ a+b & 1 \end{vmatrix} - b\begin{vmatrix} b+c & 1 \\ a+b & 1 \end{vmatrix} + c\begin{vmatrix} b+c & 1 \\ c+a & 1 \end{vmatrix} \right\}$$

$$= \frac{1}{2}\left\{ a(c+\cancel{a}-\cancel{a}-b) - b(\cancel{b}+c-a-\cancel{b}) + c(b+\cancel{c}-\cancel{c}-a) \right\}$$

$$= \frac{1}{2}\left\{ \cancel{ac} - \cancel{ab} - \cancel{bc} + \cancel{ab} + \cancel{bc} - \cancel{ac} \right\}$$

$= 0 = RHS$ $\therefore A, B, C \rightarrow$ Collinear

$\boxed{Q.3}$ $K = ?$ area = 4 sq. units.

(ii) $(-2,0)$, $(0,4)$, $(0,K)$
$\quad A \qquad B \qquad C$

$|\pm 4| = 4$

area of $\Delta ABC = 4$

$$\Rightarrow \left| \underbrace{\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & K & 1 \end{vmatrix}}_{\pm 4} \right| = 4$$

C_1 के along expand

$$\Rightarrow \frac{1}{2}\left\{ -2\begin{vmatrix} 4 & 1 \\ K & 1 \end{vmatrix} - \cancel{0|\;|}^{0} + \cancel{0|\;|}^{0} \right\} = \pm 4$$

$$\Rightarrow \left\{-2\begin{vmatrix} 4 & 1 \\ k & 1\end{vmatrix}\right\} = \pm 8$$

$$\Rightarrow \{-2\,(4-k)\} = \pm 8$$

$$\Rightarrow -8 + 2k = \boxed{\pm 8}$$

$$\Rightarrow 2k = \boxed{\pm 8} + 8$$

(+)

$2k = +8+8$

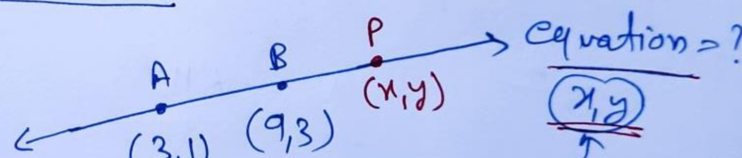
$\boxed{k = 8}$

(−)

$2k = -\cancel{8} + \cancel{8} = 0$

$\boxed{k = 0}$

Q. 4 (ii) Find equation of line joining (3,1) and (9,3) using determinants.



A, B, P → Collinear

ar(ΔABP) = 0

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

by expanding along R_3

$$\Rightarrow \left\{ x\begin{vmatrix} 1 & 1 \\ 3 & 1\end{vmatrix} - y\begin{vmatrix} 3 & 1 \\ 9 & 1\end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 9 & 3\end{vmatrix}\right\} = 0$$

$(1-3)$ $(3-9)$ $(9-9)$

$$\Rightarrow x(-2) - y(-6) + 1(0) = 0$$

$$\Rightarrow -2x + 6y = 0 \longrightarrow \boxed{2x = 6y}$$

$x = 3y$

Minors | Cofactors | DETERMINANTS

उपसारणिक सहखण्ड सारणिक

Minors : Minor of an element $\boxed{a_{ij}}$ of a determinant is the determinant obtained by deleting i^{th} row and j^{th} column (in which a_{ij} lies) $\Longrightarrow$ Denoted by (M_{ij})

e.g. $|A| = \begin{vmatrix} 2 & 5 & 6 \\ 3 & 8 & -1 \\ 0 & \boxed{7} & 3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times 3}$

Minor of $\underset{(2)}{a_{11}} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_{2\times 2}$

$= \begin{vmatrix} 8 & -1 \\ 7 & 3 \end{vmatrix}$

Minor of $\underset{(7)}{a_{32}} = M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 3 & -1 \end{vmatrix}$

Note: Determinant $\rightarrow n \times n \rightarrow$ order $= n$

minor $\longrightarrow$ ~~order~~ order $= (n-1)$

Cofactor (सहखण्ड) (A_{ij} या C_{ij})

Element $\rightarrow a_{ij}$

Minor $\rightarrow M_{ij}$

Cofactor $\rightarrow A_{ij}$

$+|\,|$

$\pm|\,|$

$(-1)^{i+j} = \pm 1$

Cofactor $= A_{ij} = (-1)^{i+j} \cdot M_{ij}$

(Cofactor of a_{ij}) (minor of a_{ij})

e.g. $|A| = \begin{vmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$(-1)^2 = 1$

cofactor of $a_{11} = A_{11} = (-1)^{1+1} \cdot M_{11}$

$(-1)^3 = -1$

$= + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of $a_{12} = A_{12} = (-1)^{1+2} \cdot M_{12}$

$= - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$= -(a_{21} \cdot a_{33} - a_{31} \cdot a_{23})$

Note

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \; R_1$$

Expand along R_1

$$\Delta = \underbrace{\underbrace{(-1)^{1+1}}_{+} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} + \underbrace{\underbrace{(-1)^{1+2}}_{-} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} + \underbrace{(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

$$= (-1)^{1+1} \cdot a_{11} \cdot M_{11} + (-1)^{1+2} \cdot a_{12} \cdot M_{12} + (-1)^{1+3} \cdot a_{13} \cdot M_{13}$$

$$= \cancel{a_{11} \cdot (G_{11}}$$

$$\Delta = a_{11} \cdot (A_{11}) + a_{12} (A_{12}) + a_{13} \cdot (A_{13})$$

element Cofactor

R₂ के along.

$$\Delta = \underbrace{a_{21} A_{21}} + \underbrace{a_{22} A_{22}} + \underbrace{a_{23} A_{23}}$$

Note: If elements of a row are multiplied by cofactors of any other row, then their sum is zero.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Cofactor = A

Row

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \Delta$$

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

Row$_1$

Row$_3$

Exercise 4.4   Minors & Cofactors (Det.)

[Q.1] (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}_{2\times 2}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ $(-1)^{i+j} \cdot M_{ij}$

(i) Element (a_{ij}) minor (M_{ij}) Cofactor (A_{ij})

$a_{11} = 2$, $M_{11} = |3|_{1\times 1} = 3$, $A_{11} = (-1)^{1+1} M_{11} = M_{11} = |3| = 3$

$a_{12} = -4$, $M_{12} = |0| = 0$, $A_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$= -(|0|) = 0$

$a_{21} = 0$, $M_{21} = |-4|_{1\times 1} = -4$, $A_{21} = (-1)^{2+1} \cdot M_{21}$

Det.

$= -M_{21}$

$= -(-4) = 4$

$a_{22} = 3$, $M_{22} = |2|_{1\times 1} = 2$, $A_{22} = (-1)^{2+2} \cdot M_{22}$

$= +M_{22}$

$= 2$

$\begin{vmatrix} + & - \\ - & + \end{vmatrix}_{2\times 2}$

$\boxed{Q.2}$ (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Part (ii) $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}_{3\times3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Elements

$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10+1 = 11$, $A_{11} = (-1)^{1+1} \cdot \underbrace{M_{11}}_{+ \cdot 11} = 11$

$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 = 6$, $A_{12} = (-1)^{1+2} \cdot M_{12} = -6$

$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3-0 = 3$, $A_{13} = (-1)^{1+3} \cdot M_{13} = 3$

$\boxed{Q.3}$ Using cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$\boxed{Q.4}$ Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Q. 4

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ - & - & a_{23} \\ - & - & a_{33} \end{vmatrix}$$

3rd Column $\quad C_3$

$$\Delta = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33}$$

~~Eleter~~ Elements — Cofactors.

$a_{13} = yz$

$a_{23} = zx$

$a_{33} = xy$

Cofactors $A_{13} = (-1)^{1+3} \cdot M_{13}$

$= (-1)^4 \cdot \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$

$= z - y$

$A_{23} = (-1)^{2+3} \cdot M_{23}$

$= - \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix}$

$= -(z - x)$

$= (x - z)$

$A_{33} = (-1)^{3+3} M_{33}$

$= + \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$

$= (y - x)$

$$\Delta = a_{13} A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33}$$

$= yz(z-y) + zx(x-z) + xy(y-x)$

$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$= z^2(y-x) - z(y^2 - x^2) + xy(y-x)$

$= z^2(y-x) - z(y-x)(y+x) + xy(y-x)$

$= (y-x)\{z^2 - z(y+x) + xy\}$

$= (y-x) \cdot \{\underline{z^2 - zy} - \underline{zx + xy}\}$

$= (y-x) \cdot \{z(z-y) - x(z-y)\}$

$= (y-x) \cdot (z-y)(z-x)$

$= + (x-y)(y-z)(z-x)$

$x \to y$, $y \to z$, $z \to x$

Q.5 (D)

$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Adjoint	adj(A)

(सहखण्डज)

Inverse	A^{-1}

(व्युत्क्रम आव्यूह)

Adjoint of a Matrix

Matrix = $A = [a_{ij}]_{n\times n}$

element

$$adj(A) = [A_{ij}]^T = [A_{ji}]$$

Cofactor

(3×3) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a_{11} का Cofactor $= A_{11}$

$$= (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

(2×2) $\rightarrow B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

$A_{11} = +a_{22}$

$$adj(B) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Trick. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula list

(1) $A \cdot adj(A) = adj(A) \cdot A = |A| I$

$I \downarrow$ Unit matrix (Identity matrix)

(2) Singular matrix $|A| = 0$ (not invertible)

Non Singular matrix $|A| \neq 0$ (Invertible)

(3) $A, B \rightarrow$ non singular $|A| \neq 0, |B| \neq 0$

$AB, BA \rightarrow$ non singular

(4) $|AB| = |A| \cdot |B|$ $\quad |ABC| = |A|\,|B|\,|C|$

(5) $|adj(A)| = |A|^{n-1}$

$n \rightarrow$ order of A

$A \rightarrow n \times n$

★ (6) $A^{-1} = \dfrac{adj\,A}{|A|}$

Inverse of A

Bonus Formulas

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

$I^{-1} = I$ $\quad AI = A$

$A \cdot A^{-1} = A^{-1} \cdot A = I$

$|KA| = K^n |A|$

$n \rightarrow$ order of A

e.g. Find inverse of matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj} A}{|A| \to 1}$$

$$|A| = \left| \begin{matrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{matrix} \right| \begin{matrix} 1 & 3 \\ 1 & 4 \\ 1 & 3 \end{matrix} = (16 + \cancel{9} + 9) - (12 + \cancel{9} + 12) = 25 - 24 = 1 \neq 0$$

(Sarrus method)

$|A| = 1 \neq 0$ (Nonsingular)

$A \to$ invertible matrix

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$$

$A_{11} = 16 - 9 = 7$

$A_{12} = -(4-3) = -1$

$A_{13} = (3-4) = -1$

$A_{21} = -(12-9) = -3$

$A_{22} = (4-3) = 1$

$A_{23} = -(3-3) = 0$

$A_{31} = (9-12) = -3$

$A_{32} = -(3-3) = 0$

$A_{33} = (4-3) = 1$

$$\text{adj}(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \frac{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}{1}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Exercise 4.5 [Determinants]

$\boxed{Q.3}$

$$A(\text{adj } A) = (\text{adj } A)A = |A|\, I$$

$$A = \begin{bmatrix} 3 & 3 \\ -4 & -6 \end{bmatrix}$$

~~Sing~~ Sign change

interchange

$$\text{adj}(A) = \begin{bmatrix} -6 & -3 \\ 4 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$A(\text{adj } A) = (\text{adj } A)\, A = |A| I$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} -6 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ -4 & -6 \end{bmatrix} = \begin{vmatrix} 3 & 3 \\ -4 & -6 \end{vmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -18+12 & \cancel{-9}+\cancel{9} \\ 2\cancel{4}-2\cancel{4} & 12-18 \end{bmatrix} = \begin{bmatrix} -18+12 & -1\cancel{8}+1\cancel{8} \\ 1\cancel{2}-1\cancel{2} & 12-18 \end{bmatrix} = \begin{pmatrix} -18 \\ +12 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = (-6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

matrix.

$$\Rightarrow \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$$

Q.4 $\quad A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}_{3\times 3}$ $\quad A(\text{adj}A) = (\text{adj}A)A = |A|I$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{ij} \rightarrow$ elements.

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$A_{ij} \rightarrow$ Cofactor $= ?$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = 8$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$$

$$\text{adj}(A) = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\underline{A(\text{adj}A)} = (\text{adj}A)A = \underline{|A|I}$$

$A.(\text{adj}A)$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} . \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$|A| = \begin{vmatrix} -1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} + 0 - 0$$

$(-1)^{1+2}$

$C_2 \to$ expansion

$$= +\{9+2\} = 11$$

$A\,(\mathrm{adj}\,A) = |A|\,I \longrightarrow I \to 3\times 3$ (Identity matrix)

$$\Rightarrow \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$\boxed{Q.5}$ Find inverse (if it exists)

$$A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} \qquad |A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 - (-8) = 14 \neq 0$$

(invertible)

$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|}$$

$$\mathrm{adj}(A) = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}}{14} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

Q.11

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} \qquad A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= 1 \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} - 0 \,|\ | + 0 \,|\ |$$

0 0

$$= (-\cos^2\alpha) - (\sin^2\alpha)$$

$$= -(\cos^2\alpha + \sin^2\alpha) = -1 = |A| \neq 0$$

invertible

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T \qquad \underset{(A_{ij})}{\text{Cofactor}} = (-1)^{i+j} \underbrace{\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}}_{\text{minor}}$$

$A_{11} = +(-\cos^2\alpha - \sin^2\alpha) = -1$

$A_{12} = -(0)$

$A_{13} = +(0)$

$A_{21} = -(0)$

$A_{22} = +(-\cos\alpha)$

$A_{23} = -(\sin\alpha)$

$A_{31} = +(0)$

$A_{32} = -(\sin\alpha)$

$A_{33} = +(\cos\alpha)$

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}}{-1}$$

Exercise-4.5 $\boxed{Q12, Q13, Q14}$ Determinants

$\boxed{Q.12}$ Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$.

Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$|AB| = |A||B| = 1 \times (-2)$
$= -2$

$|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$
$= 15 - 14 = 1$

$|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix}$
$= 54 - 56$
$= -2$

$$\text{LHS} = (AB)^{-1} = \frac{\operatorname{adj}(AB)}{|AB|} = \frac{\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}}{-2}$$

$$\text{RHS} = B^{-1} \cdot A^{-1}$$

$$= \frac{\operatorname{adj} B}{|B|} \times \frac{\operatorname{adj} A}{|A|}$$

$$= \frac{\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}}{-2} \times \frac{\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}}{1} = \frac{\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}}{-2}$$

$$= \frac{\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}}{-2} = \text{LHS}$$

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

Q.13 $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ To Prove $A^2 - 5A + 7I = 0$

$A^{-1} = ?$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

$$A^2 = A\cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

LHS $= A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2} = 0 = \text{RHS}$$

Zero matrix

Given, $(A^2 - 5A + 7I) = 0$

(multiply A^{-1} to both sides.)

$\Rightarrow (A^2 - 5A + 7I)\cdot A^{-1} = 0\cdot A^{-1}$

$\Rightarrow A\cdot\underbrace{A\cdot A^{-1}} - 5\underbrace{A\cdot A^{-1}} + 7I\cdot A^{-1} = 0$ ← zero matrix.

$\Rightarrow \underbrace{A\cdot I - 5I} + 7A^{-1} = 0$

$\Rightarrow 7A^{-1} = -A + 5I = -\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 7A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}}$ ✓

$\boxed{Q.14}$ $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $a, b = ?$

$\boxed{A^2 + aA + bI = 0}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 = A.A$

$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} = A^2$

$\boxed{A^2 + aA + bI = 0}$

$\Rightarrow \begin{bmatrix} 11 & \boxed{8} \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & \boxed{2} \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & \boxed{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \boxed{11+3a+b} & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} \boxed{0} & 0 \\ 0 & 0 \end{bmatrix}$

By comparison,

$4 + a = 0$

$\Rightarrow \boxed{a = -4}$

$3 + a + b = 0$

$\Rightarrow 3 + (-4) + b = 0$

$\Rightarrow -1 + b = 0$

$\Rightarrow \boxed{b = 1}$

Check

$11 + 3a + b = 0$

$11 - 12 + 1 = 0$ ✓

$8 + 2a = 0$

$8 - 8 = 0$ ✓

Exercise - 4.5 (Determinants)

$\boxed{Q.15}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that
$A^3 - 6A^2 + 5A + 11I = 0$

$\boxed{A^{-1} = ?}$

Identity matrix (3×3)

Zero matrix (3×3)

$A^2 = A \cdot A$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+1+2 & \cancel{1}+2-\cancel{1} & 1-\cancel{3}+\cancel{3} \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & \cancel{2}-\cancel{2}-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 33 & -13 & 58 \end{bmatrix}$

To Prove $A^3 - 6A^2 + 5A + 11I = 0$

LHS = $A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 33 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$+ 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{Zeromatrix}$$

or

RHS.

Given. $A^3 - 6A^2 + 5A + 11I = 0$ $A^{-1} = ?$

by multiplying A^{-1} to both sides.

$\Rightarrow (A^3 - 6A^2 + 5A + 11I)A^{-1} = (0)A^{-1}$

$\Rightarrow \underbrace{A^2 . A . A^{-1}} - 6A . \underbrace{A . A^{-1}} + 5\underbrace{A . A^{-1}} + 11 I . A^{-1} = 0$

$\Rightarrow A^2 . I - 6AI + 5I + 11 A^{-1} = 0$

$\Rightarrow 11 . A^{-1} = -A^2 + 6A - 5I$

$$\Rightarrow 11 . A^{-1} = -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow 11 . A^{-1} = \begin{bmatrix} \boxed{-3} & \boxed{4} & \boxed{5} \\ \boxed{6} & \boxed{-1} & \boxed{-4} \\ \boxed{5} & \boxed{-3} & \boxed{-1} \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 6 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$ ✓

$\boxed{Q.17}$ $A \to$ order 3×3 $n =$ order $= 3$

$|adj\ A| = |A|^{n-1}$

$= |A|^{3-1}$

$= |A|^2$ option (B)

$\boxed{Q.18}$ $A \to$ invertible matrix of order '2' $n=2$

$det. (A^{-1}) = |A^{-1}|$

$|A| =$ number $= k$

$adj(A) \to$ matrix

$= \left| \frac{adj(A)}{|A|} \right|$

$\boxed{\begin{array}{l} \text{Property} \\ |kA| = k^n |A| \\ n = \text{order of } A \end{array}}$

$= \left(\frac{1}{|A|}\right)^n |adj\ A|$

$= \frac{1}{|A|^n} \cdot |A|^{n-1} = \frac{1}{|A|}$

$|A^{-1}| = \frac{1}{|A|}$

$= \frac{1}{det(A)}$ option (B)

(Determinants)

Consistent System → atleast one Solution

$a_1x+b_1y=c_1$
$a_2x+b_2y=c_2$

Unique solution

Infinite no. of Solutions

Inconsistent System: No solution

Solution of system of linear equations using Inverse of a matrix

Equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2 \Rightarrow \boxed{AX=B}$$
$$a_3x+b_3y+c_3z=d_3 \Rightarrow A^{-1}AX=A^{-1}B$$
$$\Rightarrow \boxed{X=A^{-1}.B}$$

Matrix $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times3}$ (Coefficients), matrix $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$ (Variables)

matrix $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times1}$ (Constants)

$$A^{-1}=\frac{\text{adj}A}{|A|}..$$

Solution $\boxed{X=\underset{3\times3}{A^{-1}}\ \underset{3\times1}{B}}=\begin{bmatrix} \vdots \end{bmatrix}_{3\times1}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\boxed{AX=B} \longrightarrow \boxed{X = A^{-1}B}$ Solution.

$A^{-1} = \dfrac{\operatorname{adj}(A)}{|A|}$

$A^{-1} \rightarrow$ unique

$|A|$

$|A| \neq 0$

(Unique solution)

Consistent System

(O = ओ = Zero matrix)

$|A| = 0$

$(\operatorname{adj} A) \cdot B \neq O$

No solution

Inconsistent System

$(\operatorname{adj} A) \cdot B = O$

Infinite solutions

Consistent

No solution

Inconsistent

Solution of System of linear Equations using inverse of a matrix

3 Equations
3 Variables

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\boxed{AX = B}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times 1}$$

2 Equations
2 Variables

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$\boxed{AX = B}$$

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}_{2\times 2}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1}$$

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times 1}$$

$$\boxed{AX = B}$$

$$\Rightarrow A^{-1}AX = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow \boxed{X = A^{-1}B}$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$|A| \neq 0$

Comparison $\longrightarrow$ solution

x, y, z

e.g. Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Ans. We have to write all equations in the form of $AX = B$

where $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = 3(2-3) + 2(4+4) + 3(-6-4)$

$$= -3 + 16 - 30 = \underbrace{-17}_{} = |A| \neq 0$$

$$\text{adj} A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$A_{11} = +(2-3) = -1$, $A_{12} = -(8)$, $A_{13} = +(-10)$

$A_{21} = -(5)$, $A_{22} = -6$, $A_{23} = 1$

$A_{31} = -1$, $A_{32} = 9$, $A_{33} = 7$

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$|A| = -17$$

Solution $X = A^{-1}B$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}}{-17} \quad , \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

Solution

$$X = A^{-1} . B$$

$$X = -\frac{1}{17} \underbrace{\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}}_{3\times 3} . \underbrace{\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}}_{3\times 1}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17} \underbrace{\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}}_{3\times 1}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}}$$

Exercise (4.6) $\boxed{\text{DETERMINANTS}}$

Q. 1

$x + 2y = 2$
$2x + 3y = 3$

$\searrow$

$AX = B \qquad X = A^{-1}B$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad A^{-1} = \dfrac{\text{adj}A}{|A|}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$|A| \neq 0 \quad \therefore$ Consistent

- $|A| =$
  - $|A| \neq 0$ — Consistent
  - $|A| = 0$
    - $(\text{adj}A)\cdot B \neq 0$ — inconsistent
    - $(\text{adj}A)\cdot B = 0$
      - Cons.
      - incons.

Q.3

$x + 3y = 5$
$2x + 6y = 8 \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}_{2\times 2}$

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

$|A| = 0$

$\searrow$

inconsistent

$\text{adj}(A) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$\text{adj}(A)\cdot B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$= \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$

$\boxed{Q.5}$ $3x - y - 2z = 2$

$2y - z = -1 \longrightarrow AX = B$

$3x - 5y = 3$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$$

$$= \underline{-15} + \underline{3 + 12} = 0 = |A|$$

$\boxed{(\text{adj} A) \cdot B = ?}$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T \qquad A_{ij} = (-1)^{i+j} \left| \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix} \right|$$

Cofactor   minor

$A_{11} = -5$, $A_{12} = -3$, $A_{13} = -6$

$A_{21} = 10$, $A_{22} = 6$, $A_{23} = 12$

$A_{31} = 5$, $A_{32} = 3$, $A_{33} = 6$

$$\text{adj}(A) = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}^T = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$(\mathrm{adj}\,A)\cdot B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{pmatrix} -\cancel{10} - \cancel{10} + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 18 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ -6 \end{pmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\neq 0$

↑

Zero matrix.

$|A| = 0$

But $(\mathrm{adj}\,A)\cdot B \neq 0$ $\Big\}$ → inconsistent

Exercise 4.6 (Determinants)

Q.7 Solve using Matrix Method

$5x+2y=4$
$7x+3y=5$ $\Rightarrow$ $\underset{2\times 2}{\begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix}x\\y\end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix}4\\5\end{bmatrix}}$

$\boxed{A \cdot X = B}$

$AX = B$

left multiply by A^{-1}

$\Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow I.X = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1}B}$

$\Rightarrow X = \begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix} \cdot \begin{bmatrix}4\\5\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}12-10\\-28+25\end{bmatrix}_{2\times 1}$

$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$

$A^{-1} = \frac{adj\,A}{|A|}$, $A = \begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}_{2\times 2}$

$adj(A) = \begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}$

$|A| = \begin{vmatrix}5 & 2\\7 & 3\end{vmatrix} = 15-14$
$= 1$

$A^{-1} = \frac{adj\,A}{|A|} = \frac{\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}}{1}$

Comparison $\boxed{x=2,\ y=-3}$

(11) $$\left.\begin{aligned} 2x+y+z &= 1 \\ x-2y-z &= 3/2 \\ 0x+3y-5z &= 9 \end{aligned}\right\} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

$$\boxed{A \cdot X = B}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$$

$$\Rightarrow \boxed{X = A^{-1} \cdot B} \text{ Solution}$$

$$A^{-1} = \frac{\text{adj}A}{|A|}$$

$$|A| = 2(10+3) - 1(-5+0) + 1(3+0)$$

$$|A| = \cancel{2(13)}\ 26 + 5 + 3 = 34$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

Cofactor $\oplus$ $\ominus$ minor

$A_{11} = +(13)$, $A_{12} = -(-5) = 5$, $A_{13} = 3$

$A_{21} = 8$, $A_{22} = -10$, $A_{23} = -6$

$A_{31} = 1$, $A_{32} = 3$, $A_{33} = -5$

$$\text{adj}(A) = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}^T = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$X = A^{-1}.B$$

$$\Rightarrow X = \frac{\text{adj } A}{|A|} \cdot B = \frac{\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}_{3\times 3}}{34} \cdot \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow X = \frac{1}{34} \cdot \begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix}_{3\times 1} = \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix} \Rightarrow \begin{matrix} x = 1 \\ y = \frac{1}{2} \\ y = -3/2 \end{matrix}$$

Comparison

Exercise (4.6) (DETERMINANTS)

Q.15 $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$

$\downarrow$

$A^{-1} = \frac{\operatorname{adj} A}{|A|}$

$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} \begin{matrix} 2 & -3 \\ 3 & 2 \\ 1 & 1 \end{matrix} = (-8 + 12 + 15) - (10 - 8 + 18)$

$\boxed{|A| = -1} \neq 0$

$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Cofactor

$A_{ij} = (-1)^{i+j} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$

minor

$A_{11} = 0$ $A_{12} = -(-2) = 2$ $A_{13} = 1$

$A_{21} = -1$ $A_{22} = -9$ $A_{23} = -5$

$A_{31} = 2$ $A_{32} = 23$ $A_{33} = 13$

$\operatorname{adj} A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}}{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\boxed{A \cdot X = B}$$

$$AX = B$$
$$\Rightarrow A^{-1}AX = A^{-1}B$$
$$\Rightarrow \boxed{X = A^{-1} \cdot B}$$

$$\Rightarrow X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 \cancel{+} 25 + 39 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1$
$y = 2$
$z = 3$

Q.16 Price (per kg) (₹)

Equations (ATQ)

Onion → x

Wheat → y

Rice → z

$$\begin{cases} 4x + 3y + 2z = 60 \\ 2x + 4y + 6z = 90 \\ 6x + 2y + 3z = 70 \end{cases}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$A \cdot X = B$$

$$\Rightarrow A^{-1} \cdot AX = A^{-1} \cdot B$$

$$\Rightarrow \boxed{X = A^{-1} \cdot B}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

Cofactor minor

$A_{11} = +0$ $A_{12} = -(-30) = +30$ $A_{13} = -20$

$A_{21} = -5$ $A_{22} = 0$ $A_{23} = 10$

$A_{31} = 10$ $A_{32} = -20$ $A_{33} = 10$

$$\text{adj } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} \begin{matrix} 4 & 3 \\ 2 & 4 \\ 6 & 2 \end{matrix}$$

$$\begin{array}{r} 116 \\ -\ 66 \\ \hline 50 \end{array}$$

$$= (\cancel{48} + 108 + 8) - (\cancel{48} + 48 + 18)$$

$$= 50$$

$$X = A^{-1} B$$

$$x = \frac{\text{adj} A}{|A|} \cdot B = \frac{\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \cdot \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}}{50}$$

$$x = \frac{1}{50} \begin{bmatrix} 0 - 450 + 700 \\ 1800 + 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$\left.\begin{array}{l} \text{onion} \rightarrow x = 5 \\ \text{wheat} \rightarrow y = 8 \\ \text{rice} \rightarrow z = 8 \end{array}\right\}$$

Miscellaneous Exercise on Chapter 4 (DETERMINANTS)

Q.1 Prove that $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

θ ×

Ans.

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$(-1)^{i+j}$ a_{ij}

By expanding along R_1

$$\Delta = x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta)$$
$$+\cos\theta(-\sin\theta + x\cos\theta)$$

$$\Delta = -x^3 - x + x\sin^2\theta + \cancel{\sin\theta\cos\theta}$$
$$- \cancel{\sin\theta\cos\theta} + x\cos^2\theta$$

$$\Delta = -x^3 - x + x(\underbrace{\sin^2\theta + \cos^2\theta}_{1})$$

$$\Delta = -x^3 \cancel{-x} + \cancel{x}$$

$$\boxed{\Delta = -x^3}$$ Δ is independent of θ.

Miscellaneous Exercise on chapter-4 (Determinants)

Q.2 Prove without expanding

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$LHS = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$ (Using Properties)

C_3 में से (abc) Common

$$= abc \begin{vmatrix} a & a^2 & 1/a \\ b & b^2 & 1/b \\ c & c^2 & 1/c \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

(Distribute)

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ c^2 & 1 & c^3 \end{vmatrix}$$

$C_2 \leftrightarrow C_3$ (interchange)

= Again by $C_2 \leftrightarrow C_1$

$$= +\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \underline{RHS}$$

Miscellaneous Exercise on chapter ④ (DETERMINANTS)

Q.3 Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

By expanding along C_3

$$= -\sin\alpha \begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta \end{vmatrix}$$

$$- 0 . \cancel{\begin{vmatrix} & \end{vmatrix}} \to 0$$

$$+ \cos\alpha \begin{vmatrix} \cos\alpha.\cos\beta & \cos\alpha.\sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix}$$

$$= -\sin\alpha.\{-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta\}$$
$$+ \cos\alpha.\{\cos\alpha.\cos^2\beta + \cos\alpha.\sin^2\beta\}$$

$$= +\sin^2\alpha\{\underbrace{\sin^2\beta + \cos^2\beta}_{1}\}$$
$$+ \cos^2\alpha.\{\underbrace{\cos^2\beta + \sin^2\beta}_{1}\}$$

$$= \sin^2\alpha + \cos^2\alpha = 1 = \text{Ans.}$$

Miscellaneous Exercise on Chapter 4 (Determinants)

Q.4, Q.5

Q.4 If a, b, c are real number, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$

Show that either $a+b+c=0$ or $a=b=c$.

Ans. Given

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Properties

$R_i \to R_i \pm kR_j$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & (b+c) & c+a \end{vmatrix} = 0$$

C_1 में से $2(a+b+c)$ Common

$$\Rightarrow 2(a+b+c) \cdot \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 0$$

Operation $\boxed{R_2 \to R_2 - R_1}$ & $\boxed{R_3 \to R_3 - R_1}$

$$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} = 0$$

expansion C_1

by expanding along 'C_1'

$$\Rightarrow 2(a+b+c)\cdot\left\{1\begin{vmatrix} b-c & c-a \\ b-a & c-b\end{vmatrix} - \underbrace{0|\ |}_{0} + \underbrace{0|\ |}_{0}\right\} = 0$$

$$\Rightarrow 2(a+b+c)\cdot\{(b-c)\cdot(c-b) - (b-a)(c-a)\} = 0$$

$$\Rightarrow 2(a+b+c)\cdot\left[bc - b^2 - c^2 + \cancel{bc} - \cancel{bc} + ab + ac - a^2\right] = 0$$

$$\Rightarrow \underbrace{2(a+b+c)}_{0}\cdot\underbrace{\left[a^2+b^2+c^2-ab-bc-ca\right]}_{0} = 0$$

$\boxed{a+b+c=0}$ or $\boxed{a^2+b^2+c^2-ab-bc-ca=0}$

$$2a^2+2b^2+2c^2-2ab-2bc-2ca=0$$

squar

$\boxed{(\ldots)^2 \geq 0}$

0 → 0
1
2 +
3

$$\Rightarrow (a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)=0$$

$$\Rightarrow \underbrace{(a-b)^2}_{0}+\underbrace{(b-c)^2}_{0}+\underbrace{(c-a)^2}_{0}=0$$

$(\ )^2 \geq 0$

$(\ )^2$
↓
player
Batsman

$(a-b)^2=0$ ⇓

$\Rightarrow a-b=0$ $\boxed{b=c}$ & $\boxed{c=a}$ $\Rightarrow \boxed{a=b=c}$

$\boxed{a=b}$ &

Q.5 Solve the equation $\quad x = ?$

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0 \quad , \quad a \neq 0$$

Given
$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Properties Operation $\boxed{C_1 \rightarrow C_1 + C_2 + C_3}$

$$\Rightarrow \begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$$

Common

$$\Rightarrow (3x+a) \begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$$

$\boxed{R_2 \rightarrow R_2 - R_1}$ & $\boxed{R_3 \rightarrow R_3 - R_1}$

$$\Rightarrow (3x+a) \cdot \begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$$

expand along C_1

$$\Rightarrow (3x+a) \cdot \left\{ 1(a^2 - 0) - \underset{0}{\cancel{0|\;|}} + \underset{0}{\cancel{0|\;|}} \right\} = 0$$

$$\Rightarrow (3x+a) \cdot a^2 = 0 \Rightarrow 3x + a = 0 \Rightarrow \boxed{x = -\frac{a}{3}} \quad a \neq 0$$

Miscellaneous Exercise on Chapter 4 [DETERMINANTS]

Q.6 $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$ Prove

LHS = $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$ $4 = 2\times 2$

C_1 a C_2 b C_3 c Common

$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$

$\boxed{C_1 \to C_1 + C_2 + C_3}$

$= abc \begin{vmatrix} 2a+2c & c & a+c \\ 2a+2b & b & a \\ \underbrace{2b+2c}_{\text{2 Common}} & b+c & c \end{vmatrix}$

$= 2abc \begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ \underbrace{b+c}_{C_1} & \underbrace{b+c}_{C_2} & c \end{vmatrix}$

$C_1 \to C_1 - C_2$

$$= 2abc \begin{vmatrix} a & c & a+c \\ a & b & a \\ 0 & b+c & c \end{vmatrix}$$

$$\boxed{R_2 \to R_2 - R_1}$$

$$= 2abc \begin{vmatrix} a & c & a+c \\ 0 & b-c & -c \\ 0 & b+c & c \end{vmatrix}$$

expand

$$= 2abc \cdot \left\{ a \begin{vmatrix} b-c & -c \\ b+c & c \end{vmatrix} - 0 \,|\;| + 0 \,|\;| \right\}$$

$$= 2a^2bc \cdot \left(bc - \cancel{c^2} + bc + \cancel{c^2} \right)$$

$$= (2a^2bc) \times (2bc)$$

$$= 4a^2b^2c^2 = RHS$$

Miscellaneous Exercise on Chapter-(4) DETERMINANTS

Q7, Q8 → Hint only ☺
I
Fully solve

Q.7 If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

$A^{\times}$ $\quad A^{-1}$ $\qquad (A^{-1})^{-1} = A$

B $\quad \cancel{B^{-1}}$

Ans. $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

Property of Invertible matrices.

? $\quad$ Given

$B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $\qquad B^{-1} = \dfrac{\text{adj } B}{|B|} \rightarrow (1)$

$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ -1 & 3 \\ 0 & -2 \end{matrix}$ $\qquad (\searrow + \searrow + \searrow) - (\nearrow + \nearrow + \nearrow)$

$= (3 + 0 - 4) - (0 + 0 - 2)$

$= -1 + 2 = 1 = |B|$

$\text{adj } B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$ $\qquad A_{ij} = (-1)^{i+j} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$

Cofactor $\quad \pm \quad$ minor

$$B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} A_{11} = 3, & A_{12} = 1, & A_{13} = 2 \\ A_{21} = 2, & A_{22} = 1, & A_{23} = 2 \\ A_{31} = 6, & A_{32} = 2, & A_{33} = 5 \end{matrix}$$

$$\text{adj}(B) = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj}\, B}{|B|} = \frac{\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}}{1}$$

$|B| = 1$

$$(AB)^{-1} = B^{-1} \cdot A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$\rightarrow \cdot \downarrow$

$$= \begin{bmatrix} 9 - \cancel{30} + \cancel{30} & -3 + \cancel{12} - \cancel{12} & 3 - 10 + 12 \\ 3 - 15 + 10 & -1 + 6 - 4 & \cancel{1} - \cancel{5} + \cancel{4} \\ 6 - 30 + 25 & -\cancel{2} + \cancel{12} - 10 & 2 - \cancel{10} + \cancel{10} \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = B^{-1} \cdot A^{-1}$$

Q.8 लगे रहो ____

Miscellaneous Exercise on Chapter 4 $\boxed{\text{DETERMINANTS}}$

$\boxed{Q.9}$ + $\boxed{Q.10}$

$\boxed{Q.9}$ Evaluate
$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Properties

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$2x+2y = 2(x+y)$

$\boxed{C_1 \rightarrow C_1 + C_2 + C_3}$

$$\Delta = \begin{vmatrix} 2x+2y & y & x+y \\ 2x+2y & x+y & x \\ 2x+2y & x & y \end{vmatrix}$$

common

$$= 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix}$$

$\boxed{R_2 \rightarrow R_2 - R_1}$ & $\boxed{R_3 \rightarrow R_3 - R_1}$

$$= 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$$

$C_1 \rightarrow$ expand

$= 2(x+y) \cdot \left\{ 1 \begin{vmatrix} x & -y \\ (x-y) & -x \end{vmatrix} - 0 + 0 \right\}$

$= 2(x+y) \cdot (-x^2 + xy - y^2)$

$= -2(x+y)(x^2 - xy + y^2)$

$= -2(x^3 + y^3)$

$\boxed{Q.10}$

$\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$

$R_2 \to R_2 - R_1$ $R_3 \Rightarrow R_3 - R_1$

$= \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$

expand along c_1

$= 1 \begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} - 0 + 0$

0 0

$= xy - 0$

$= xy$

Miscellaneous Exercise on Chapter 4 $\boxed{\text{DETERMINANTS}}$

$\boxed{\text{Q11, Q12, Q13}}$

Q.11

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

$$\text{LHS} = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} \quad \text{by } C_3 \to C_3 + C_1$$

C_1 C_3

$$= \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix} = (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

$C_3 \to$ Common $(\alpha+\beta+\gamma)$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix} \begin{matrix} \\ \to R_2 \text{ में } (\beta-\alpha) \text{ common} \\ \to R_3 \text{ में } (\gamma-\alpha) \text{ Common} \end{matrix}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ 1 & \beta+\alpha & 0 \\ 1 & \gamma+\alpha & 0 \end{vmatrix} \to C_3 \text{ के along expand}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \cdot \left\{ 1 \begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\alpha \end{vmatrix} - 0 \begin{vmatrix} \ \end{vmatrix} + 0 \begin{vmatrix} \ \end{vmatrix} \right\}$$

0 0

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \cdot (\gamma+\cancel{\alpha}-\beta-\cancel{\alpha})$$

$$= (\alpha+\beta+\gamma) \cdot (\beta-\alpha)(\gamma-\alpha) \cdot (\gamma-\beta)$$

$\ominus$ ✓ $\ominus$

$(+)$

$$= (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma) = RHS$$

$\boxed{Q.12}$ $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$

$$LHS = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$ by property

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \begin{array}{l} \to R_1 \to x \\ \to R_2 \to y \\ \to R_3 \to z \end{array}$$

C_3 से 'p' common

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Firstly $C_2 \leftrightarrow C_3$
Secondly $C_1 \leftrightarrow C_2$ interchange

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1+pxyz) \begin{array}{c} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \end{array} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

$\boxed{R_2 \to R_2 - R_1}$ & $\boxed{R_3 \to R_3 - R_1}$

$$= (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \begin{array}{l} \\ \to (y-x) \text{ common} \\ \to (z-x) \text{ common} \end{array}$$

$$= (1+pxyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

expand C_1

$$= (1+pxyz)(y-x)(z-x) \cdot \left\{ 1(z+\cancel{x}-y-\cancel{x}) - 0(\ ) + 0(\ ) \right\}$$

$$= [1+pxyz](y-x)(z-x)(z-y)$$

$$= (1+pxyz)(x-y)(y-z)(z-x) = RHS$$

$\boxed{Q.13}$ $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c).(ab+bc+ca)$

LHS = $\begin{vmatrix} \boxed{3a} & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

$\boxed{C_1 \to C_1 + C_2 + C_3}$

$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

C_1 में $(a+b+c)$ Common.

$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$ $(\ )-(\ )$

C_1 के along expand.

$= (a+b+c).\{1.(4bc+2ab+2ac+\cancel{a^2}-\cancel{a^2}+ab+ac-bc)\}$

$= (a+b+c)\{3ab+3bc+3ca\} = 3(a+b+c)(ab+bc+ca)$

RHS

Miscellaneous Exercise on Chapter 4 (DETERMINANTS)

Q14, Q15

$\boxed{Q.14}$ $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$ (Prove)

$$\text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

$\boxed{R_2 \to R_2 - 2R_1}$ & $\boxed{R_3 \to R_3 - 3R_1}$

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

By expanding along C_1

$$= 1 \cdot \begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 \begin{vmatrix} \end{vmatrix} + 0 \begin{vmatrix} \end{vmatrix}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \quad\quad\quad 0$

$$= (7+\cancel{3p}) - (6+\cancel{3p}) = 1 = \text{RHS}$$

$\boxed{Q.15}$ $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$ (Prove)

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\text{LHS} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

RHS=0

$\downarrow$ Cos(A+B) Formula

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$

Property

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix} - \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\sin\delta \end{vmatrix}$$

C_3 Common $\cos\delta$ $\qquad$ C_3 Common $\sin\delta$

$$= \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix} - \sin\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta & \sin\beta \\ \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

$C_2 = C_3$ $\qquad$ $C_1 = C_3$

Property $\rightarrow$ when 2 columns are identical then $\Delta = 0$

$$= \cos\delta.(0) - \sin\delta.(0)$$

$$= 0 - 0$$

$$= 0 = \text{RHS}$$

Miscellaneous Exercise on Chapter (4) (DETERMINANTS)

Q.16 Solve the system of equations ($x, y, z = ?$)

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

$\underbrace{\frac{1}{x} = a}_{\text{Let}}$, $\frac{1}{y} = b$, $\frac{1}{z} = c$

$$\left.\begin{aligned} 2a + 3b + 10c &= 4 \\ 4a - 6b + 5c &= 1 \\ 6a + 9b - 20c &= 2 \end{aligned}\right\} \longrightarrow \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}_{3\times 1}$$

$$A \cdot X = B$$

$$\Rightarrow A^{-1} A X = A^{-1} B$$

$$\Rightarrow I X = A^{-1} B$$

$$\Rightarrow \boxed{X = A^{-1} B}$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2(120 - 45) - 4(-60 - 90) + 6(15 + 60)$$

$$= 2(75) - 4(-150) + 6(75)$$

$$= 1200$$

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$A_{ij} = \text{Cofactor} = (-1)^{i+j} \left| \, \vdots \, \right|$ minor

$A_{11} = 75$, $A_{12} = +110$, $A_{13} = 72$

$A_{21} = 150$, $A_{22} = -100$, $A_{23} = 0$

$A_{31} = 75$, $A_{32} = 30$, $A_{33} = -24$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T$$

$$\text{adj}(A) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}}{1200}$$

$$X = A^{-1} . B$$

$$\Rightarrow X = \frac{\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}}{1200} . \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$a = \frac{1}{2} = \frac{1}{x} \Rightarrow \boxed{x=2}$$

$$b = \frac{1}{3} = \frac{1}{y} \Rightarrow \boxed{y=3}$$

$$c = \frac{1}{5} = \frac{1}{z} \Rightarrow \boxed{z=5}$$

Miscellaneous Exercise on Chapter (4) $\boxed{\text{DETERMINANTS}}$

$\boxed{\text{Q.17}}$ If a, b, c are in A.P., then the Determinant
$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$ is

(A) 0 (B) 1 (C) x (D) $2x$

Ans. $a, b, c \longrightarrow AP$

$\boxed{2b = a+c}$

$b-a = c-b$

$\Rightarrow b+b = a+c$

$2b = a+c$

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \times \frac{2}{2}$$

~~$C_2 \to C_2 - C_1$~~

$$\Delta = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 2x+6 & 2x+8 & 2x+4b \\ x+4 & x+5 & x+2c \end{vmatrix}$$ R_1, R_3

$R_2 \to R_2 - (R_1 + R_3)$

$$\Delta = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$4b-(2a+2c)$

$\Rightarrow 2\{\cancel{2b}-(\cancel{a+c})\}$

$= 0$

$\Delta = 0$ ✓

Miscellaneous Ex. on Chapter 4 DETERMINANTS

Q.18 If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is–

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \qquad A^{-1} = ?$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix} = xyz$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A_{ij} = (-1)^{i+j} \cdot \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

Cofactor minor

$\pm$

$(-1)^{\text{Even}} = +1$

$(-1)^{\text{odd}} = -1$

$A_{11} = yz$, $A_{12} = 0$, $A_{13} = 0$

$A_{21} = 0$, $A_{22} = xz$, $A_{23} = 0$

$A_{31} = 0$, $A_{32} = 0$, $A_{33} = xy$

$$\text{adj}(A) = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}^T = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

option (A)

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}}{xyz} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

Q.19 Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$.

Then

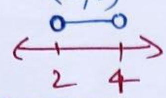
(A) Det(A) = 0 (B) Det(A) $\in (2, \infty)$

(C) Det(A) $\in (2, 4)$ (D) Det(A) $\in [2, 4]$

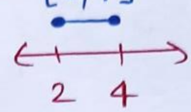
interval

Small bracket (,) $\downarrow$ not include

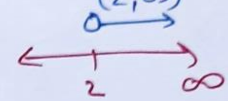
(2,4)



[2,4]



(2,∞)



Square bracket [,] $\downarrow$ include

$$\det(A) = |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$ R_1 के along expand

$$\Rightarrow |A| = 1\begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$$

$$\Rightarrow |A| = 1 + \sin^2\theta - \underbrace{\sin\theta(\cancel{-\sin\theta} + \cancel{\sin\theta})}_{0} + \sin^2\theta + 1$$

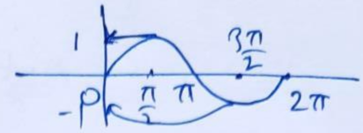
$$\Rightarrow |A| = 2 + 2\sin^2\theta$$

$|A| = 2 + 2\sin^2\theta$

$0 \le \theta \le 2\pi$

$0^\circ \le \theta \le 360^\circ$

$\boxed{-1 \le \sin\theta \le 1}$



$\Rightarrow \quad 0 \le \sin^2\theta \le 1$

Square

$(-1)^2 = +1$

$\ominus \longrightarrow \oplus$

$\sin\theta$	-1	-0.9	-0.8 ---	-0.1	0	0.1	0.2	---	0.9	1
Square $\sin^2\theta$	1	0.81	0.64	0.01	0	0.01	0.04		0.8	1

minimum

maximum

$0 \le \sin^2\theta \le 1$

$|A| = \underline{2 + 2\sin^2\theta}$

$\Rightarrow \quad 0 \le 2\sin^2\theta \le 2$

$+2 \qquad +2 \qquad +2$

$\Rightarrow \quad 2 \le \underbrace{2 + 2\sin^2\theta} \le 4$

$2 \le |A| \le 4$ inequality

$|A| \in [2, 4]$

$\det(A) \in [2, 4]$

option D