Differential Equations | अवकल समीकरण

Basic Concepts: An equation which has derivative (derivatives) of the dependent variable (like y) with respect to independent variable (like x), is called a Differential Eqⁿ.

e.g. $x\left(\frac{dy}{dx}\right) + y = 0$

e.g. $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

e.g. $\frac{d^2y}{dx^2} = 5$

Ordinary Diff. Eqⁿ.

Only one Indep. Variable

Note:

$$\frac{dy}{dx} = y' = y_1 \qquad \frac{d^3y}{dx^3} = y''' = y_3$$

$$\frac{d^2y}{dx^2} = y'' = y_2 \qquad \frac{d^ny}{dx^n} = y_n$$

Order of a differential Equation (कोटि) = (Order of Highest order derivative)

e.g. $\frac{dy}{dx} = e^x \longrightarrow$ order = 1

e.g. $\frac{d^2y}{dx^2} = -y \longrightarrow$ order = 2

e.g. $\frac{d^3y}{dx^3} = -x^2\left(\frac{d^2y}{dx^2}\right)^3 \longrightarrow$ order = 3

Degree of a Differential Equation: (घात)

→ इसके लिए equation, derivatives में Polynomial होना जरूरी है।

→ Derivatives में polynomial format होने के बाद ~~Highest~~ Highest order derivative की Highest power = Degree.

e.g. $\boxed{\frac{d^3y}{dx^3}} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$ ✓

Order = 3
Degree = 1 ✓

e.g. $\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin\sqrt{y} = 0$ ✓

Order = 1
Degree = 2

e.g. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$ → Not in Polynomial format

Order = 1
Degree = Not Defined

e.g. $\left(\frac{dy}{dx}\right)^1 + \sin x = 0$ ✓

Order = 1
Degree = 1

e.g. $x + \sin\left(\frac{dy}{dx}\right) = 0 \Rightarrow \sin\frac{dy}{dx} = -x$

$\Rightarrow \frac{dy}{dx} = \sin^{-1}(-x)$

Order = 1
Degree = 1

e.g. $y''' + y^2 + e^{(y')} = 0$ →

Order = 3
Degree = Not Defined

e.g. $y_2 + \sqrt{y_1} = 0$ ✗

$y_2 = -\sqrt{y_1}$

$(y_2)^2 = (y_1)$ ✓

Order = 2
Degree = 2

Exercise 9.1

Determine order and degree (if defined) of Differential equations.

Q.1 $\underbrace{\frac{d^4y}{dx^4}} + \sin(y''') = 0$

$\boxed{\text{Order} = 4}$

$\boxed{\text{Degree} = \text{Not Defined}}$

$y_4 + \sin(y_3) = 0$

$\Rightarrow \sin(y_3) = -y_4$

$\Rightarrow y_3 = \sin^{-1}(-y_4)$

Polynomial Format

Q.2 $\boxed{y' + 5y = 0} \Rightarrow \boxed{\frac{dy}{dx} + 5y = 0}$

order = 1

Degree = 1

Q.3 $\left(\frac{ds}{dt}\right)^4 + 3s\,\underbrace{\frac{d^2s}{dt^2}} = 0$

order = 2

Degree = 1

Q.4 $\underbrace{\left(\frac{d^2y}{dx^2}\right)^2} + \cos\left(\frac{dy}{dx}\right) = 0$

order = 2

Degree = Not Defined.

Q.5 $\left(\dfrac{d^2y}{dx^2}\right)^1 = \cos 3x + \sin 3x$ Order $= 2$
Degree $= 1$

⑥ $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ Order $= 3$
Degree $= 2$

Q.7 $y''' + 2y'' + y' = 0 \longrightarrow$ Order $= 3$
Degree $= 1$

Q.8 $y' + y = e^x \longrightarrow$ Order $= 1$
Degree $= 1$

Q.9 $y'' + (y')^2 + 2y = 0 \longrightarrow$ Order $= 2$
Degree $= 1$

H.O.D.

Q.10 $y'' + 2y' + \sin y = 0 \longrightarrow$ Order $= 2$
Degree $= 1$

Q.11 The degree of the differential equation

$\left(\dfrac{d^2y}{dx^2}\right)^3 + \left(\dfrac{dy}{dx}\right)^2 + \sin\left(\dfrac{dy}{dx}\right) + 1 = 0$ is $\rightarrow$ (A) 3 (B) 2

(C) 1 (D) Not Defined

Derivatives $\frac{d}{dx}$ Polynomial $\rightarrow$ Not a Polynomial in Deri.

Q.12 The order of the differential equation

~~$2x^2 \, \frac{d^2y}{}$~~ $\boxed{2x^2\left(\dfrac{d^2y}{dx^2}\right) - 3\dfrac{dy}{dx} + y = 0}$ is —

(A) 2 (B) 1 (C) 0 (D) Not Defined.

General & Particular Solutions of a D.E.

function

For Example.

$$\underbrace{\frac{dy}{dx} = y}$$ ← Differential Equation

General Solution $\boxed{y = ke^x}$ $k \in R$

Arbitrary Constants (स्वेच्छ अचर)

Particular Solutions → $\begin{cases} y = e^x \\ y = 2e^x \\ y = 3e^x \\ \vdots \end{cases}$

Class 11th Example

$\boxed{\sin x = 0}$

General Solution

$\boxed{x = n\pi}$, $n \in I$

Particular Solutions

$\begin{cases} x = 0 \\ x = \pi \\ x = 2\pi \\ \vdots \quad \vdots \end{cases}$

Definition. General Solution → Solution which have Arbitrary Constants $(a, b, c, c_1, \ldots)$

Particular Solutions → general solutions से मिले हुए ऐसे Solutions जिनमें Arbitrary constants की जगह कुछ खास value (number) हो।

(Free from arbitrary Constants)

e.g. Verify that the function $y = e^{-3x}$ is a solution of the differential equations $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

LHS = RHS

Solution: $y = e^{-3x}$

$y' = \frac{dy}{dx} = -3e^{-3x}$

$y'' = \frac{d^2y}{dx^2} = 9e^{-3x}$

$$LHS = y'' + y' - 6y$$
$$= 9\cancel{e^{-3x}} + (-3\cancel{e^{-3x}}) - 6(\cancel{e^{-3x}})$$
$$= 0 = RHS$$

General Solution

e.g. Verify that the function $\boxed{y = a\cos x + b\sin x}$, where $a, b \in R$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

LHS = RHS

Sol^n: $y = a\cos x + b\sin x$

$\frac{dy}{dx} = -a\sin x + b\cos x$

$\frac{d^2y}{dx^2} = -a\cos x - b\sin x$

(a, b → Arb. Const.)

$$LHS = \frac{d^2y}{dx^2} + y$$
$$= -a\cancel{\cos x} - b\cancel{\sin x}$$
$$+ a\cancel{\cos x} + b\cancel{\sin x}$$
$$= 0 = RHS$$

Note: ✯ Order of Differential Equation = Number of Arbitrary Constants in General Solution

2 = 2

order = order of Highest order derivative

$\boxed{\text{Exercise 9.2}}$

$\boxed{LHS = RHS}$

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$\boxed{Q.1}$ $y = e^x + 1$: $y'' - y' = 0$ D.E. $\to LHS = RHS$

$y' = e^x$

$y'' = e^x$

$LHS = y'' - y'$

$= e^x - e^x = 0 = RHS$

verified

$\boxed{Q.2}$ $y = x^2 + 2x + c$: $\boxed{y' - 2x - 2 = 0}$

$\Rightarrow y' = 2x + 2$

$\Rightarrow \boxed{y' - 2x - 2 = 0}$

$LHS = RHS$

$\boxed{Q.3}$ $y = \cos x + c$: $\boxed{y' + \sin x = 0}$

$\Rightarrow y' = -\sin x$

$LHS = y' + \sin x$

$= -\cancel{\sin x} + \cancel{\sin x} = 0 = RHS$

verified

$\boxed{Q.4}$ $y = \sqrt{1+x^2}$: $y' = \dfrac{xy}{1+x^2}$

Diff. $\to$ Chain Rule

$\Rightarrow y' = \dfrac{1}{\cancel{2}\sqrt{1+x^2}} \times (\cancel{2}x)$

$\Rightarrow y' = \dfrac{x}{\sqrt{1+x^2}}$.

$LHS = y' = \dfrac{x}{\sqrt{1+x^2}}$

$RHS = \dfrac{x(y)}{1+x^2} = \dfrac{x\sqrt{1+x^2}}{(1+x^2)^1} = \dfrac{x}{\sqrt{1+x^2}}$

$LHS = RHS$

Solution

Q.5 $\boxed{y = Ax}$: $xy' = y \quad (x \neq 0)$

$y' = A$

$LHS = xy'$

$= xA$

$= Ax = y = RHS$

verified

Q.6 $y = x\sin x$: $xy' = y + x\sqrt{x^2 - y^2}$

$\Rightarrow y' = 1.\sin x + x\cos x$

$\boxed{y' = \sin x + x\cos x}$

$LHS = xy' = x(\sin x + x\cos x) = x\sin x + x^2\cos x$

$RHS = y + x\sqrt{x^2 - y^2} = x\sin x + x\sqrt{x^2 - (x\sin x)^2}$

$= x\sin x + x^2\sqrt{1 - \sin^2 x}$

$= x\sin x + x^2\cos x = LHS$

Q.7 $xy = \log y + c$: $\boxed{y' = \frac{y^2}{1 - xy}}$

Diff. w.r.t. x

$\Rightarrow 1.y + xy' = \frac{1}{y}.y'$

$\Rightarrow y = \frac{y'}{y} - xy'$

$\Rightarrow y = y'\left(\frac{1}{y} - x\right)$

$\Rightarrow y = y'\left(\frac{1 - xy}{y}\right)$

$\Rightarrow \boxed{\frac{y^2}{1 - xy} = y'}$

Q.8 $y - \cos y = x$: $(y\sin y + \cos y + x).y' = y$

$\Rightarrow y' + \sin y.y' = 1$

$\Rightarrow \boxed{y'(1 + \sin y) = 1}$

$LHS = (y\sin y + \cos y + x).y'$

$= (y\sin y + \cancel{\cos y} + y - \cancel{\cos y}).y'$

$= y(\sin y + 1).y'$ $\to 1$

$= y.1 = y = RHS.$

Q.9 $x+y=\tan^{-1}y$: $y^2 y' + y^2 + 1 = 0$

Diff. w.r.t x

$\Rightarrow 1 + y' = \frac{1}{1+y^2} \cdot y'$

$\Rightarrow (1+y')(1+y^2) = y'$

$\Rightarrow 1+y^2+\cancel{y'}+y'\cdot y^2 = \cancel{y'}$

$$\boxed{y^2 \cdot y' + y^2 + 1 = 0}$$

Q.10 $y=\sqrt{a^2-x^2}$, $x+y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{a^2-x^2}}\cdot(0-\cancel{2}x)$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}} = \frac{-x}{y}$

$LHS = x + y \cdot \frac{dy}{dx}$

$= x + \cancel{y}\left(-\frac{x}{\cancel{y}}\right)$

$= x - x = 0 = RHS$

Q.11 The number of arbitrary constants in the general solⁿ of a differential equation of fourth order are:

(A) 0 (B) 2 (C) 3 (D) 4

$\boxed{\text{Order of D.E. = No. of Arbitrary Constants in General Solution.}}$

Q.12 The number of arbitrary constants in the particular solution of a differential equation of third order are —

(A) 3 (B) 2 (C) 1 (D) 0

Particular Solutions $\longrightarrow$ No Arbitrary Constants.

Formation of Differential Equation :→

e.g. General Solution (given) Eqⁿ. ⟶ Differential Equation ($x, y, \frac{dy}{dx}, y', y'' \ldots$)

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \longrightarrow \boxed{xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0}$$

Arbitrary Constants (स्वेच्छ अचर)

Note: Order of differential Equation = Number of arbitrary Constants ☆

Steps to Form a Differential Equation →

① Given general solution को उतनी बार differentiate करके रख लो, जितने उसमें Arbitrary Constants हो।

② जितनी भी equations अब आपके सामने हो, उनसे Arbitrary constants को हटा लो। Done !!!

e.g. Form the differential equation representing the family of curves $\boxed{y = a\sin(x+b)}$, where a, b are arbitrary constants.

Ans. No. of Arb. Constants = 2 (a, b) . order = 2

$y = a\sin(x+b)$ — ①

$y' = a\cos(x+b)$ — ②

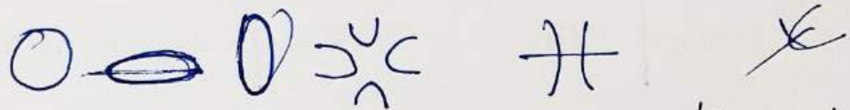
$y'' = -a\sin(x+b)$ — ③

By eqⁿ ① & ③ :→

$$\boxed{y'' = -(y)}$$

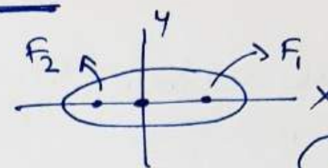
$$\boxed{y'' + y = 0}$$

$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$



e.g. Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

$(a > b)$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a, b \rightarrow$ A.C.

Ans. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —— (1)

Diff. $\Rightarrow \frac{2x}{a^2} + \frac{2y \cdot y'}{b^2} = 0$

$\Rightarrow \boxed{\frac{x}{a^2} + \frac{y \cdot y'}{b^2} = 0}$ —— (2)

Diff. $\Rightarrow \boxed{\frac{1}{a^2} + \frac{y' \cdot y'}{b^2} + \frac{y \cdot y''}{b^2} = 0}$

$\Rightarrow \frac{1}{a^2} = -\frac{(y')^2}{b^2} - \frac{y \cdot y''}{b^2}$ —— (3)

By eq^n (2) & (3) :→

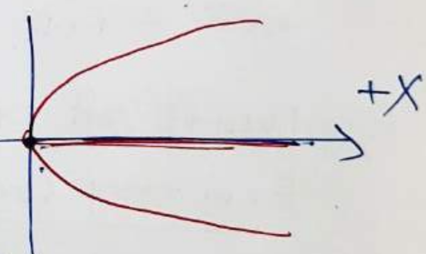
$\Rightarrow x\left(-\frac{(y')^2}{b^2} - \frac{y y''}{b^2}\right) + \frac{y \cdot y'}{b^2} = 0$

$\Rightarrow \left(-x(y')^2 - xy \cdot y'' + yy' = 0\right) -$

$\Rightarrow \boxed{xyy'' + x(y')^2 - yy' = 0}$

e.g. Form the differential eq^n. representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

Standard Form. $\boxed{y^2 = 4ax}$



a = arbitrary constant

order = 1

$y^2 = 4ax$ —— (1)

Diff.

$2y \cdot y' = 4a \cdot 1$ —— (2)

$\frac{\text{by eq}^n (1)}{\text{eq}^n (2)}$: $\frac{\cancel{y^2}\, y}{2\cancel{y} \cdot y'} = \frac{\cancel{4a}\, x}{\cancel{4a}}$

$\Rightarrow \frac{y}{2y'} = x \Rightarrow \boxed{y = 2xy'}$

$$\boxed{y = 2x \cdot \frac{dy}{dx}}$$

$\boxed{\text{Exercise 9.3}} \leftarrow$ (Formation of Differential equation)

Form a differential equation:

Q.1 $\boxed{\dfrac{x}{a}+\dfrac{y}{b}=1}$ — ① No. of Arbitrary Constants = 2 $<^{a}_{b}$

Differentiation = 2

Order = 2

Diff w.r.t. 'x'

$\Rightarrow \dfrac{1}{a}+\dfrac{y'}{b}=0$ — ②

Again Diff.

$\Rightarrow 0+\dfrac{y''}{b}=0 \Rightarrow \boxed{y''=0}$ Diff. equation.

Q.2 $y^2=a(b^2-x^2)$ — ① No. of Arb. Constants = 2 $<^{a}_{b}$

Diff. = 2

$\Rightarrow y^2=ab^2-ax^2$

by Diff. w.r.t. x →

$\Rightarrow \cancel{2}y\cdot y'=0-\cancel{2}ax$

$\Rightarrow \boxed{y.y'=(-a)x}$ — ②

Again by Diff. w.r.t. x

$\Rightarrow \boxed{y'.y'+y.y''=-a}$ — ③

By eqⁿ ② & ③ →

$\Rightarrow y.y'=\left((y')^2+y.y''\right)x$

$\Rightarrow yy'=xy.y''+x(y')^2$

$\Rightarrow \boxed{xyy''+x(y')^2-y.y'=0}$

Q3 $y = ae^{3x} + \underline{be^{-2x}}$ — (1) Arb = (2) a, b

$y' = 3ae^{3x} - 2\underline{be^{-2x}}$ — (2)

a, b

Elimination

$y'' = 9\underline{ae^{3x}} + 4\underline{be^{-2x}}$ — (3)

$$2y = 2a\underline{e^{3x}} + \cancel{2be^{-2x}} \quad (1)$$
$$+\ y' = 3a\underline{e^{3x}} - \cancel{2be^{-2x}} \quad (2)$$
$$2y + y' = 5ae^{3x}$$
$$\frac{2y+y'}{5} = \underline{a.e^{3x}}$$

$$3y = \cancel{3ae^{3x}} + 3be^{-2x} \quad (1)$$
$$y' = \cancel{3ae^{3x}} - 2be^{-2x} \quad (2)$$
$$-\quad -\quad +$$
$$3y - y' = 5be^{-2x}$$
$$\frac{3y-y'}{5} = \underline{be^{-2x}}$$

by putting the values of $a.e^{3x}$ & $b.e^{-2x}$ in eqn (3) :→

$$\Rightarrow y'' = 9\left(\frac{2y+y'}{5}\right) + 4\left(\frac{3y-y'}{5}\right)$$

$$\Rightarrow y'' = \frac{18y + 9y' + 12y - 4y'}{5}$$

$$\Rightarrow y'' = \frac{30y + 5y'}{5} \Rightarrow y'' = 6y + y'$$

$$\Rightarrow \boxed{y'' - y' - 6y = 0}$$

Q.4 $y = \underline{e^{2x}}.\underline{(a+bx)}$ — (1) No. of Arbitrary constants = (2)

$$\Rightarrow y' = 2\boxed{e^{2x}.(a+bx)} + e^{2x}.(0 + b.1)$$

$$\Rightarrow y' = 2y + be^{2x}$$

$$\Rightarrow \boxed{y' - 2y = be^{2x}} \quad (2)$$

Diff.

$$\Rightarrow \boxed{y'' - 2y' = 2be^{2x}} \quad (3)$$

By eqn (2)/(3)

$$\frac{y' - 2y}{y'' - 2y'} = \frac{\cancel{be^{2x}}}{2\cancel{be^{2x}}}$$

Cross multiply

$\Rightarrow 2y' - 4y = y'' - 2y'$

$\Rightarrow 0 = y'' - 2y' - 2y' + 4y$

$\Rightarrow \boxed{y'' - 4y' + 4y = 0}$

Q.5 $y = e^x(a\cos x + b\sin x)$ —— (1) Arb. = 2 < a, b.

Diff.

$\Rightarrow y' = \underbrace{e^x(a\cos x + b\sin x)}_{y} + e^x(-a\sin x + b\cos x)$

$\Rightarrow y' = y + e^x(-a\sin x + b\cos x)$

$\Rightarrow y' - y = e^x(-a\sin x + b\cos x)$ —— (2)

again diff.

$\Rightarrow y'' - y' = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x)$

$\Rightarrow y'' - y' = \underbrace{e^x(-a\sin x + b\cos x)}_{\text{eq}^n (2) \;\to\; y' - y} - \underbrace{e^x(a\cos x + b\sin x)}_{\text{by eq}^n (1) \;\to\; y}$

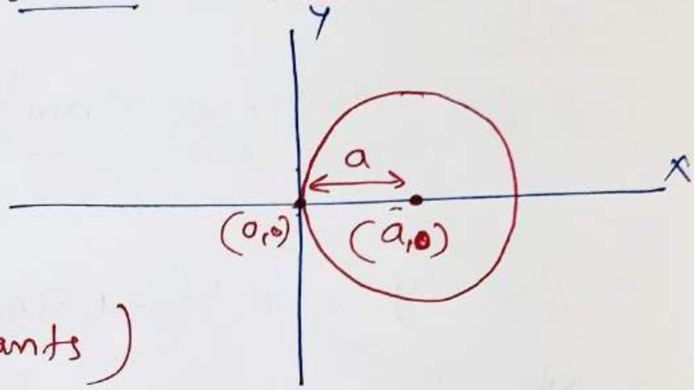
$\Rightarrow y'' - y' = (y' - y) - (y)$

$\Rightarrow y'' - y' = y' - 2y$

$\Rightarrow \boxed{y'' - 2y' + 2y = 0}$

Exercise 9.3 [Formation of Differential Equation]

Q.6 Form the differential equation of the ~~ma~~ family of circles touching the y-axis at origin.



Circle

Centre = $(a,0)$

radius = a (let)

(Arbitrary constants)

$\Rightarrow$ Eqn. $(x-a)^2+(y-0)^2=(a)^2$

$\Rightarrow \quad x^2+\cancel{a^2}-2ax+y^2=\cancel{a^2}$

$\Rightarrow \quad x^2+y^2=2ax \quad \text{—} (1)$ No. of Arb. Constants = 1

Diff. w.r.t. (x) →

$\Rightarrow \quad \cancel{2}x+\cancel{2}y.y'=\cancel{2}a$

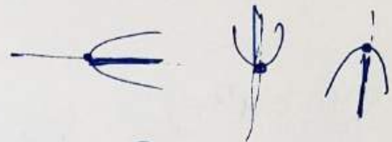
$\Rightarrow \quad x+yy'=a \quad \text{—} (2)$

by eqn (1) & (2) → $x^2+y^2=2(x+yy')x$

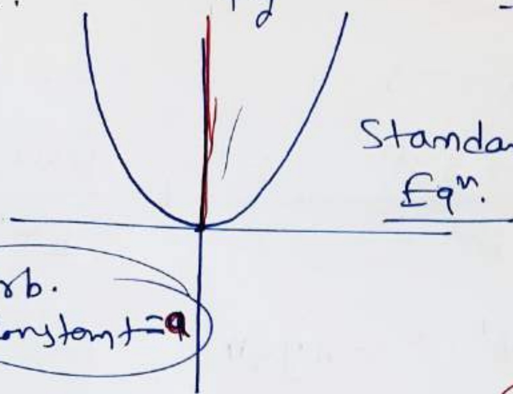
$\Rightarrow \quad x^2+y^2=2x^2+2xy.y'$

$\Rightarrow \quad \boxed{y^2=x^2+2xyy'}$

Q.7 Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.



Ans.



Standard Eqn.

Family of Parabolas

$$\boxed{x^2 = 4ay}$$

D.E. = ?

A.C.

$x^2 = 4ay$ — (1)

Arb. Constant = a

Diff.

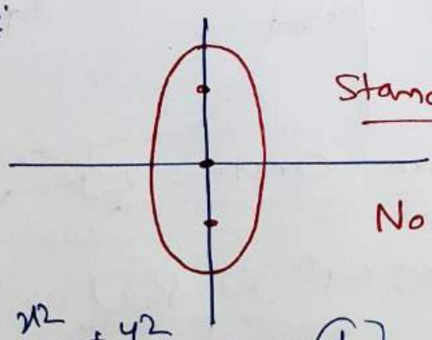
$$2x = 4a \cdot y' \quad \text{— (2)}$$

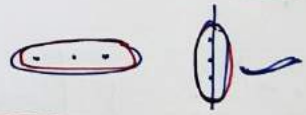
by eqn (1)/(2) $\rightarrow$ $\dfrac{x^{\cancel{2}}}{2\cancel{x}} = \dfrac{\cancel{4a}\,y}{\cancel{4a}\cdot y'} \Rightarrow \dfrac{x}{2} = \dfrac{y}{y'}$

$$\Rightarrow xy' = 2y \Rightarrow \boxed{xy' - 2y = 0}$$

Q.8 Form the differential eqn. of family of ellipses having foci on y-axis and centre at origin.

Ans.





Standard Form $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$

a, b ∈ R

No. of Arb. Constants = 2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{— (1)}$$

Diff. w.r.t. (x)

$$\Rightarrow \frac{2x}{a^2} + \frac{2y \cdot y'}{b^2} = 0$$

$$\Rightarrow \boxed{\frac{x}{a^2} + \frac{y \cdot y'}{b^2} = 0} \quad \text{— (2)}$$

Again by Diff. w.r.t. (x)

$$\frac{1}{a^2} + \frac{(y')^2 + y \cdot y''}{b^2} = 0$$

$$\Rightarrow \frac{1}{a^2} = -\left(\frac{(y')^2 + y \cdot y''}{b^2}\right)$$

— (3)

$$\boxed{\frac{x}{a^2}+\frac{yy'}{b^2}=0} \quad \text{— (2)} \qquad \boxed{\frac{1}{a^2} = -\frac{(y')^2+yy''}{b^2}} \quad \text{— (3)}$$

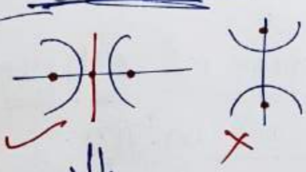
by eqn. (2) & (3) →

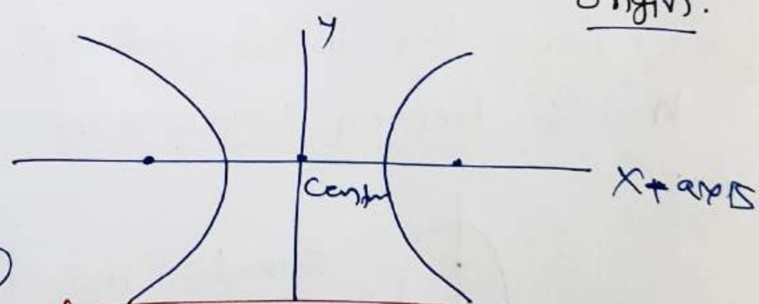
$$\Rightarrow x\cdot\left(-\frac{(y')^2+y.y''}{b^2}\right)+\frac{yy'}{b^2}=0$$

$$\Rightarrow -\frac{x(y')^2-xyy''}{\cancel{b^2}}+\frac{yy'}{\cancel{b^2}}=0$$

$$\Rightarrow \boxed{xy.y''+x(y')^2-yy'=0}$$ D.E.

(Q.9) Form the differential equation of the family of Hyperbolas having foci on x-axis and centre at origin.





No. of Arb. Const. = 2 → a, b

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \quad \text{— (1)}$$

Diff. w.r.t. (x)

$$\Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\cdot y'=0$$

$$\Rightarrow \boxed{\frac{x}{a^2}=\frac{y.y'}{b^2}} \quad \text{— (2)}$$

Diff. w.r.t. (x)

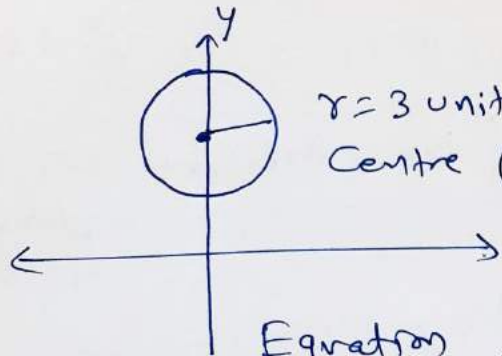
$$\Rightarrow \frac{1}{a^2}=\frac{(y')^2+y.y''}{b^2} \quad \text{— (3)}$$

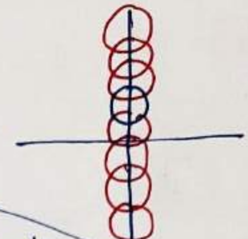
by eqn (2) & (3)

$$\Rightarrow x\left(\frac{(y')^2+y.y''}{\cancel{b^2}}\right)=\frac{yy'}{\cancel{b^2}}$$

$$\Rightarrow \boxed{xyy''+x(y')^2-yy'=0}$$

Q.10 Form the Differential equation of the family of circles having centre on y-axis and radius 3 units.





No. of arb. constants = 1

Equation $(x-0)^2 + (y-b)^2 = (3)^2$

$$\Rightarrow \boxed{x^2 + (y-b)^2 = 9} \quad \text{---} (1)$$

diff. w.r.t. x →

$$\Rightarrow 2x + 2(y-b)\cdot y' = 0$$

$$\Rightarrow x + (y-b)\cdot y' = 0 \Rightarrow \boxed{(y-b) = \frac{-x}{y'}} \quad \text{---} (2)$$

by eqn (1) & (2) :→

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9 \Rightarrow x^2 + \frac{x^2}{(y')^2} = 9$$

$$\Rightarrow (x^2-9) + \frac{x^2}{(y')^2} = 0 \Rightarrow \left\{\frac{(x^2-9)\cdot(y')^2 + x^2}{(y')^2}\right\} = 0$$

$$\Rightarrow \boxed{(x^2-9)\cdot(y')^2 + x^2 = 0}$$

$\boxed{Q.11}$ Which of the following differential eqⁿ. has $\boxed{y = c_1 e^{x} + c_2 e^{-x}}$ as the general solution ?

(A) $y'' + y = 0$ (B) $y'' - y = 0$ (C) $y'' + 1 = 0$ (D) $y'' - 1 = 0$

$y = c_1 e^{x} + c_2 e^{-x}$ — (1)

$y' = c_1 e^{x} - c_2 e^{-x}$ — (2)

$y'' = c_1 e^{x} + c_2 e^{-x}$ — (3)

$\boxed{\text{No. of Arb. Const.} = 2}$ → c_1, c_2

by eqⁿ (1) & (3) →

$\Rightarrow y'' = y$

$\boxed{y'' - y = 0}$ ✓

$\boxed{Q.12}$ Which of the following differential equation has $\boxed{y = x}$ as one of its particular solution ?

(A) $y'' - x^2 \cdot y' + xy = x$ (B) $y'' + xy' + xy = x$

(C) $y'' - x^2 y' + xy = 0$ (D) $y'' + xy' + xy = 0$

$x \rightarrow x$ ✓

$y \rightarrow x$ ✓

$y' \rightarrow 1$ ✓

$y'' \rightarrow 0$ ✓

Check (A) $0 - \cancel{x^2} \cdot 1 + \cancel{x^2} = x$

$0 \neq x$ ✗

(B) → $0 + \cancel{x} + x^2 = \cancel{x}$

$x^2 \neq 0$ ✗

$\boxed{\text{(C)} \quad 0 - \cancel{x^2} + \cancel{x^2} = 0 \quad \Rightarrow \quad 0 = 0}$ ✓

Methods of Solving Differential Equations

Differential Equations with Variables Separable	Homogeneous Differential Equations	Linear Differential Equations

Differential Equations with Variables Separable ← Today

Variable Separation method

- First, separate the variables. (x, dx वाले एक तरफ, y, dy वाले एक तरफ)
- Second, integrate.

e.g. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{(1+y^2)(1+x^2)}$.

$\int \text{—} \cdot dx$

Ans by Variable Separation:

$$\Rightarrow (1+y^2)dy = \frac{1}{1+x^2} \cdot dx$$

by integrating,

$$\Rightarrow \int (1+y^2) \cdot dy = \int \frac{1}{1+x^2} \cdot dx$$

$$\Rightarrow \boxed{y + \frac{y^3}{3} = \tan^{-1} x + C}$$

Arbitrary Constant

General Solution.

e.g. Find the equation of the curve passing through the point (1,1) whose differential equation is $x\,dy = (2x^2+1)\cdot dx$, $(x \neq 0)$.

Ans. $x\,dy = (2x^2+1)\cdot dx$

by variable separation.

$$\Rightarrow dy = \left(\frac{2x^2+1}{x}\right)\cdot dx$$

by integrating

$$\Rightarrow \int dy = \int \left(2x + \frac{1}{x}\right)\cdot dx$$

$$\Rightarrow \boxed{y = x^2 + \log|x| + c}$$

Curve

this curve passes through the point (1,1)

$$\Rightarrow \cancel{1} = \cancel{1} + \cancel{\log(1)} + c$$

$$\Rightarrow \boxed{0 = c}$$

Particular curve:

$$\boxed{y = x^2 + \log|x|}$$

e.g. In a bank, Principal increases continuously at the rate of 5% per year. In how many years ₹1000 double itself.

Principal = P

Rate of change of Principal $= \frac{d(P)}{dt} = 5\%$ of 'P'

$$\Rightarrow \frac{dP}{dt} = \frac{\cancel{5}}{\cancel{100}_{20}} \times P \leftarrow \text{Differential eq}^n.$$

(t, P)

$$\Rightarrow \frac{dP}{dt} = \frac{P}{20}$$

$$\Rightarrow \frac{dP}{P} = \frac{dt}{20}$$

Integrate

$$\Rightarrow \int \frac{dP}{P} = \int \frac{dt}{20} \Rightarrow \log|P| = \frac{1}{20}t + c$$

P = +ve

$$\boxed{\log(P) = \frac{t}{20} + C}$$

Present	$P=1000$	$t=0$ years
Future	$P=2000$	T ?? years
	↑ P	↑ t ?

Present: $P=1000$

$t=0$

Put

$$\log(1000) = 0 + C \Rightarrow C = \log(1000)$$

Update

$$\boxed{\log(P) = \frac{t}{20} + \log(1000)}$$

Future $\begin{cases} P=2000 & \text{Double of } 1000 \\ t=T \end{cases}$

$$\Rightarrow \log(2000) = \frac{T}{20} + \log(1000)$$

$$\Rightarrow \log(2000) - \log(1000) = \frac{T}{20}$$

$\log m - \log n = \log\frac{m}{n}$

$$\Rightarrow \log\left(\frac{2000}{1000}\right) = \frac{T}{20}$$

$$\Rightarrow T = 20 \cdot \log_e(2) \text{ years}$$

$$\Rightarrow T = 20 \times (0.6931)$$

$\log_e 2 = 0.6931$

$$T \approx 13.8 \text{ years}$$

Exercise 9.4 — Variable Separation Method

Q.1 $\dfrac{dy}{dx} = \dfrac{1-\cos x}{1+\cos x}$ — Find the General Solution ($y = ?$)

(Variable Separation)

$1-\cos x = 2\sin^2\frac{x}{2}$

$1+\cos x = 2\cos^2\frac{x}{2}$

$\Rightarrow dy = \dfrac{1-\cos x}{1+\cos x}\cdot dx$

by integration:

$\Rightarrow \int dy = \int \dfrac{1-\cos x}{1+\cos x}\cdot dx$

$\Rightarrow \int dy = \int \dfrac{\cancel{2}\sin^2\frac{x}{2}}{\cancel{2}\cos^2\frac{x}{2}}\cdot dx$

$\Rightarrow \int 1.dy = \int \tan^2\frac{x}{2}\cdot dx$

$\Rightarrow y = \int\left(\sec^2\frac{x}{2} - 1\right)\cdot dx$

$y = \dfrac{\tan\frac{x}{2}}{\left(\frac{1}{2}\right)} - x + C$

$\Rightarrow \boxed{y = 2\tan\frac{x}{2} - x + C}$

General solⁿ.

Q.2 $\dfrac{dy}{dx} = \sqrt{4-y^2}$ $\quad(-2 < y < 2)$

$\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

$\Rightarrow \dfrac{dy}{\sqrt{4-y^2}} = dx$

Integration.

$\Rightarrow \int \dfrac{dy}{\sqrt{4-y^2}} = \int dx$

$\Rightarrow \sin^{-1}\left(\dfrac{y}{2}\right) = (x + c)$

$\Rightarrow \dfrac{y}{2} = \sin(x+c)$

$\Rightarrow \boxed{y = 2\sin(x+c)}$

Q.3 $\dfrac{dy}{dx} + y = 1 \quad (y \neq 1)$

$\int \frac{1}{x} \cdot dx = \log|x| + c$

$\Rightarrow \dfrac{dy}{dx} = (1-y)$

V.S. method

$\Rightarrow \dfrac{dy}{(1-y)} = dx$

integrate:

$\Rightarrow \int \dfrac{dy}{1-y} = \int dx$

$\Rightarrow -\int \dfrac{dy}{y-1} = \int dx$

$\Rightarrow \int \dfrac{dy}{y-1} = -\int dx$

$\Rightarrow \log(y-1) = (-x + c)$

$\Rightarrow (y-1) = e^{-x+c}$

$\Rightarrow y = 1 + e^{-x+c}$

$\Rightarrow y = 1 + e^{-x} \cdot e^{c}$

$e^{c} \rightarrow$ A, these const. $= A$

$\Rightarrow \boxed{y = 1 + A \cdot e^{-x}}$

Q.4 $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow \sec^2 x \cdot \tan y \cdot dx = -\sec^2 y \cdot \tan x \cdot dy$

$\Rightarrow \dfrac{\sec^2 x \cdot dx}{\tan x} = -\dfrac{\sec^2 y \cdot dy}{\tan y}$

integration $\rightarrow$

$\Rightarrow \int \dfrac{\overbrace{\sec^2 x \cdot dx}^{dt}}{\underbrace{\tan x}_{t}} = -\int \dfrac{\sec^2 y \cdot dy}{\tan y}$

Let $\tan x = t$

$\sec^2 x \cdot dx = dt$

$\int \dfrac{dt}{t}$

$= \log(t) + c$

$\Rightarrow \log(\tan x) = -\log(\tan y) + c$

$\Rightarrow \cancel{\log x = -\log y + c}$

$\Rightarrow \log(\tan x) = -\log(\tan y) + c$

$\Rightarrow \log(\tan x) + \log(\tan y) = c$

$\Rightarrow \log(\tan x \cdot \tan y) = c$

$\Rightarrow \tan x \cdot \tan y = e^{c} = c_1$ New Constant

$$\boxed{\tan x \cdot \tan y = c_1}$$

$\boxed{Q.5}$ $(e^{x} + e^{-x}) \cdot dy - (e^{x} - e^{-x}) \cdot dx = 0$

$\Rightarrow (e^{x} + e^{-x}) \cdot dy = (e^{x} - e^{-x}) \cdot dx$

$\Rightarrow dy = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \cdot dx$

integrate →

$\Rightarrow \int dy = \int \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \cdot dx$ → dt → t

Let

$e^{x} + e^{-x} = t$

$(e^{x} - e^{-x}) \cdot dx = dt$

$\Rightarrow y = \int \dfrac{dt}{t}$

$\Rightarrow y = \log(t) + c$

$$\Rightarrow \boxed{y = \log(e^{x} + e^{-x}) + c}$$

Q.6 $\dfrac{dy}{dx} = (1+x^2).(1+y^2)$

$\Rightarrow \dfrac{dy}{1+y^2} = (1+x^2).dx$

integrate $\rightarrow$

$\Rightarrow \displaystyle\int \frac{dy}{1+y^2} = \int (1+x^2).\,dx$

$\Rightarrow \boxed{\tan^{-1} y = x + \frac{x^3}{3} + C}$

Q.7 $y\log y\,dx - x\,dy = 0$

$\Rightarrow y\log y.dx = x\,dy \Rightarrow \dfrac{dx}{x} = \dfrac{dy}{y.\log y}$

$\Rightarrow$ int. $\displaystyle\int \frac{dx}{x} = \int \frac{dy}{y.\log y}$

$\Rightarrow \log x = \log(\log y) + c$

$\Rightarrow x = e^{\log(\log y) + c}$

$\Rightarrow x = e^{\log_e(\log y)} . e^c$

$\Rightarrow x = \log y . e^c$

$\Rightarrow \dfrac{x}{e^c} = \log y$

$\Rightarrow e^{x/e^c} = y \Rightarrow \boxed{e^{xK} = y.}$

$\displaystyle\int \frac{dy}{y.(\log y)}$ $dy \rightarrow dt$, $\log y \rightarrow t$

let $\log y = t$

$\Rightarrow \frac{1}{y}.dy = dt$

$\Rightarrow \displaystyle\int \frac{dt}{t} = \log t + c$

$= \log(\log y) + c$

$a^{\log_a b} = b$

$\dfrac{1}{e^c} = K$

$\boxed{Q.8}$ $x^5 \frac{dy}{dx} = -y^5 \Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$

by int^n,

$$\Rightarrow \int \frac{dy}{y^5} = -\int \frac{dx}{x^5}$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$$

$$\Rightarrow \int y^{-5} \cdot dy = -\int x^{-5} \cdot dx$$

$$\Rightarrow \frac{y^{-5+1}}{-5+1} = -\left(\frac{x^{-5+1}}{-5+1}\right) + C$$

$$\Rightarrow \left\{\left(\frac{y^{-4}}{-4}\right) = -\left(\frac{x^{-4}}{-4}\right) + C\right\} \times -4$$

$$\Rightarrow y^{-4} = -x^{-4} - 4C$$

$$\Rightarrow x^{-4} + y^{-4} = -4C = C_1 = \text{New Constants}$$

$$\boxed{x^{-4} + y^{-4} = C_1}$$

$\boxed{Q.9}$ $\frac{dy}{dx} = \sin^{-1}x \Rightarrow dy = \sin^{-1}x \cdot dx$

ILATE

by integration $\rightarrow$ $\int dy = \int \underset{\text{I}}{\sin^{-1}x} \cdot \underset{\text{II}}{1} \cdot dx$

$$\boxed{\int \text{I} \cdot \text{II} = \text{I} \int \text{II} - \int\left(\text{I}' \cdot \int \text{II}\right)}$$

int. by Parts.

$$\Rightarrow y = \sin^{-1}x \cdot (x) - \int \left(\frac{1}{\sqrt{1-x^2}} \cdot x\right) \cdot dx$$

$$\Rightarrow y = x \cdot \sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Subs. $1 - x^2 = t$

$$\Rightarrow -2x \cdot dx = dt \Rightarrow \boxed{x\,dx = \frac{dt}{-2}}$$

$\Rightarrow y = x\sin^{-1}x - \frac{1}{-2}\int \frac{dt}{\sqrt{t}}$

$\Rightarrow y = x\sin^{-1}x + \frac{1}{\cancel{2}}(\cancel{2}\sqrt{t}) + c$

$\Rightarrow \boxed{y = x\sin^{-1}x + \sqrt{1-x^2} + c}$

$\int \frac{1}{\sqrt{t}} \cdot dt$

$\int t^{-1/2} \cdot dt$

$\frac{t^{-1/2+1}}{-1/2+1} = 2\sqrt{t}$

$\boxed{Q.10}$ $e^x \tan y \, dx + (1-e^x) \cdot \sec^2 y \cdot dy = 0$

$\Rightarrow e^x \tan y \cdot dx = (e^x - 1) \cdot \sec^2 y \cdot dy$

$\Rightarrow \frac{e^x \cdot dx}{e^x - 1} = \frac{\sec^2 y \cdot dy}{\tan y}$

integration.

$\Rightarrow \int \frac{e^x \cdot dx \rightarrow dt}{e^x - 1 \rightarrow t} = \int \frac{\sec^2 y \cdot dy \rightarrow dm}{\tan y \rightarrow m}$

$\Rightarrow \log(e^x - 1) = \log(\tan y) + c$

$\Rightarrow e^{\left(\log(e^x-1) - c\right)} = \cancel{e}^{\left(\cancel{\log}(\tan y)\right)}$

$\Rightarrow \cancel{e}^{\log(e^x-1)} \cdot e^{-c} \rightarrow K = \tan y$

$\Rightarrow \boxed{(e^x - 1) \cdot K = \tan y}$ ✓

Exercise 9.4

Variable Separation Method.

Find a particular solution satisfying the given Condition →

Q.11 $(x^3+x^2+x+1)\cdot\frac{dy}{dx} = 2x^2+x$; $y=1$ when $x=0$.

$\Rightarrow dy = \frac{2x^2+x}{x^3+x^2+x+1}\cdot dx$

integration.

$\Rightarrow \int dy = \int \frac{2x^2+x}{x^3+x^2+x+1}\cdot dx$

$\Rightarrow y = \int \frac{2x^2+x}{x^2(x+1)+1(x+1)}\cdot dx$

$y = \int \frac{2x^2+x}{(x+1)(x^2+1)}\cdot dx$

Partial fraction

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Proper Method: $2x^2+x = A(x^2+1) + (Bx+C)\cdot(x+1)$

Coeff. Comparison. $\Rightarrow (2x^2+x) = Ax^2+A + Bx^2+Bx + Cx+C$

$x^2 \to 2 = A+B$ — (1)

$x \to 1 = B+C$ — (2)

Constant $\to 0 = A+C$ — (3)

$B = 3/2$

$A = \frac{1}{2}$

$C = -\frac{1}{2}$

$$y = \int \frac{2x^2+x}{(x+1)(x^2+1)} \cdot dx \Rightarrow y = \int \left(\frac{1/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right) \cdot dx$$

$$\Rightarrow y = \frac{1}{2} \int \frac{1}{x+1} \cdot dx + \frac{3}{2 \cdot 2} \int \frac{\overbrace{2x \, dx}^{dt}}{\underbrace{x^2+1}_{t}} \; \frac{1}{2} \int \frac{1}{x^2+1} \cdot dx$$

$$\Rightarrow \boxed{y = \frac{1}{2} \log(x+1) + \frac{3}{4} \cdot \log(x^2+1) - \frac{1}{2} \tan^{-1} x + c}$$

$x=0, \; y=1$

$$\Rightarrow 1 = \frac{1}{2} \underbrace{\cancel{\log(1)}}_{0} + \frac{3}{4} \underbrace{\cancel{\log(1)}}_{0} - \frac{1}{2} \underbrace{\cancel{\tan^{-1}(0)}}_{0} + c$$

$$\boxed{1 = c}$$

Curve $y = \frac{1}{2} \log(x+1) + \frac{3}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1}(x) + 1$

$$\Rightarrow y = \frac{1}{4} \left\{ 2 \log(x+1) + 3 \cdot \log(x^2+1) \right\} - \frac{\tan^{-1} x}{2} + 1$$

$\boxed{n \cdot \log m = \log m^n}$

$$\Rightarrow y = \frac{1}{4} \left\{ \log(x+1)^2 + \log(x^2+1)^3 \right\} - \frac{\tan^{-1} x}{2} + 1$$

$\log m + \log n = \log mn$

$$\Rightarrow \boxed{y = \frac{1}{4} \log\left[(x+1)^2 \cdot (x^2+1)^3\right] - \frac{\tan^{-1} x}{2} + 1}$$

(Q.12) $x(x^2-1)\frac{dy}{dx} = 1$; $y = 0$ when $x = 2$

$\Rightarrow dy = \frac{dx}{x(x^2-1)} \rightarrow$ Integrate.

$\Rightarrow \int dy = \int \frac{dx}{x(x^2-1)} = \int \frac{dx}{x(x+1)(x-1)}$

Partial Fraction

$\Rightarrow y = \int \left(\frac{-1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1} \right) \cdot dx$

$x \neq 0$, $\neq 0$, $\neq 0$

$$\Rightarrow \boxed{y = -1 \cdot \log x + \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-1) + c}$$

$\Rightarrow y = \frac{1}{2}\left\{ -2\log x + \log(x+1) + \log(x-1) \right\} + c$

$\Rightarrow y = \frac{1}{2}\left[\log \frac{(x+1)(x-1)}{x^2} \right] + c$

General soln.

$$\Rightarrow \boxed{y = \frac{1}{2}\log\left(\frac{x^2-1}{x^2}\right) + c} \longrightarrow \boxed{x = 2, y = 0}$$

$0 = \frac{1}{2}\log\left(\frac{3}{4}\right) + c$

$c = -\frac{1}{2}\log\left(\frac{3}{4}\right)$

Particular soln.

$\Rightarrow y = \frac{1}{2}\log\left(\frac{x^2-1}{x^2}\right) - \frac{1}{2}\log\left(\frac{3}{4}\right)$

$\boxed{Q. 13}$ $\cos\left(\frac{dy}{dx}\right) = a$; $(a \in R)$; $\underline{y=1 \text{ when } x=0}$

$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$

$\Rightarrow dy = \underline{\cos^{-1} a} \cdot dx$ Integration.

$\Rightarrow \int dy = \cos^{-1}(a) \cdot \int dx \Rightarrow \boxed{y = \cos^{-1} a \cdot (x) + c}$

$\underline{x=0, y=1}$

$\Rightarrow 1 = \underbrace{\cos^{-1} a}_{0} \cdot (0) + c \Rightarrow \boxed{c=1}$

Particular solⁿ: $y = \cos^{-1} a \cdot x + 1$

$\Rightarrow y - 1 = \cos^{-1} a \cdot x$

$\Rightarrow \left(\frac{y-1}{x}\right) = \cos^{-1} a \Rightarrow \boxed{\cos\left(\frac{y-1}{x}\right) = a}$

$\boxed{Q.14}$ $\frac{dy}{dx} = y \tan x$; $\underline{y=1 \text{ when } x=0}$.

$\Rightarrow \frac{dy}{y} = \tan x \cdot dx$ Integration

$\Rightarrow \int \frac{dy}{y} = \int \tan x \cdot dx \Rightarrow \boxed{\log(y) = \log \sec x + C}$

$\underline{x=0, y=1}$ $\underline{(0,1)}$

$\underbrace{\log(1)}_{\to 0} = \underbrace{\log(\underbrace{\sec 0}_{1})}_{\to 0} + C. \Rightarrow \boxed{C=0}$

Particular solⁿ: $\log y = \log \sec x$

$\Rightarrow \boxed{y = \sec x}$

Q. 15 Find the equation of a curve passing through the point (0,0) and whose differential equation is $y' = e^x \sin x$.

Ans. $\frac{dy}{dx} = e^x \sin x \Rightarrow dy = e^x \sin x \, dx$

(Integration)

$\Rightarrow \int dy = \int e^x \sin x \, dx$ — II, I — ILATE

(integration by Parts)

$\boxed{\int \mathrm{I} \cdot \mathrm{II} = \mathrm{I} \int \mathrm{II} - \int \left(\mathrm{I}' \cdot \int \mathrm{II} \right)}$

$\Rightarrow \cancel{y}$

$\Rightarrow y = \sin x \cdot e^x - \int \cos x \cdot e^x \cdot dx$ — I, II — ILATE

$\Rightarrow y = \sin x \cdot e^x - \left[\cos x \cdot e^x + \left(\int \sin x \cdot e^x \cdot dx \right) \right]$ → y

$\Rightarrow y = \sin x \cdot e^x - \cos x \cdot e^x - y + C$

$\Rightarrow \boxed{2y = e^x (\sin x - \cos x) + C}$

(0, 0) → x, y; C → 1

$0 = e^0 (0 - 1) + C$

$\Rightarrow 0 = 1(-1) + C \Rightarrow 0 = -1 + C \Rightarrow \boxed{1 = C}$

$$\boxed{2y = e^x (\sin x - \cos x) + 1}$$

Exercise 9.4 (Variable Separation method)

Q.16 For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point $(1,-1)$.

C

Ans. $xy\frac{dy}{dx} = (x+2).(y+2)$

$\Rightarrow \frac{y\,dy}{y+2} = \frac{(x+2)}{x}dx$ (Integration)

$\Rightarrow \int \frac{y}{y+2}.dy = \int \frac{x+2}{x}.dx$

$\Rightarrow \int \frac{y+2-2}{(y+2)}.dy = \int \left(1+\frac{2}{x}\right).dx$

$\Rightarrow \int\left(1-\frac{2}{y+2}\right).dy = x + 2\log(x) + c$

$\Rightarrow \boxed{y - 2\log(y+2) = x + 2\log x + c}$ Curve.

Passes through $(1,-1)$

$x=1,\ y=-1$

$-1-2\log(1)^{\,0} = 1+2\log 1^{\,0} + c$

$\boxed{-2 = c}$

Again eqn (curve) $y - 2\log(y+2) = x + 2\log x - 2$

$\Rightarrow y - x + 2 = \log(x^2) + \log(y+2)^2$

$\Rightarrow \boxed{y - x + 2 = \log\left(x^2.(y+2)^2\right)}$

Q.17 Find the equation of a Curve passing through the Point (0, -2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

AOD
Slope of tangent $= \frac{dy}{dx}$

Ans. ATQ.

$$\frac{dy}{dx} \times y = x$$

$\Rightarrow$ $y.dy = x\,dx$

integration

$\Rightarrow$ $\int y\,dy = \int x\,dx$

$\Rightarrow$ $\boxed{\frac{y^2}{2} = \frac{x^2}{2} + c}$

$c \rightarrow 2$

Passes through (0, -2)

$$\frac{(-2)^2}{2} = \frac{0}{2} + c$$

$\Rightarrow$ $\boxed{2 = c}$

Curve $\left(\frac{y^2}{2} = \frac{x^2}{2} + 2\right) \times 2$

$\Rightarrow$ $y^2 = x^2 + 4$

$\Rightarrow$ $\boxed{y^2 - x^2 = 4}$

Q.18 At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact (-4, -3) to the point. Find the equation of the curve given that it passes through (-2, 1).

$m = \frac{y_2 - y_1}{x_2 - x_1}$

(m)
(x, y) $(-4, -3)$

Ans. ATQ $\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$

$\Rightarrow$ $\frac{dy}{y+3} = 2\frac{dx}{x+4}$

$\Rightarrow$ $\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$

$\boxed{\log(y+3) = 2\log(x+4) + c}$

$(-2, 1)$

$\Rightarrow \log(4) = 2\log(2) + c$

$\Rightarrow \log(4) = \log(4) + c$

$\boxed{0 = c}$

$\Rightarrow \log(y+3) = 2\log(x+4) + 0$

$\Rightarrow \cancel{\log}(y+3) = \cancel{\log}(x+4)^2$

$\Rightarrow \boxed{(y+3) = (x+4)^2}$

अरे।

Q.19 The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after 't' seconds.

Ans. $\begin{pmatrix}\text{Rate of change}\\ \text{in volume (V)}\end{pmatrix}$ = Constant (K)

$t=0$	$r=3$
$t=3$	$r=6$
$t=t$	$r=?$

$\Rightarrow \frac{dv}{dt} = K \leftarrow$ D.E.

$\Rightarrow dv = K \cdot dt$

$\boxed{V = \frac{4}{3}\pi r^3}$

integration

$\Rightarrow \int dv = K\int dt$

$\Rightarrow \boxed{V = Kt + C}$

$\Rightarrow \boxed{\frac{4}{3}\pi r^3 = Kt + C}$

$t=0, r=3$ Put

$\Rightarrow \frac{4}{3}\pi(3^3) = K(0) + C$

$\Rightarrow \boxed{36\pi = C}$

updated $\boxed{\frac{4}{3}\pi r^3 = Kt + 36\pi}$

$t=3, r=6$ put

$\Rightarrow \frac{4}{3}\pi 6^3 = K(3) + 36\pi$

$\Rightarrow 8 \times 36\pi = 3K + 36\pi$

$\Rightarrow 8 \times 36\pi - 36\pi = 3K$

$\Rightarrow 96\pi - 12\pi = K$

$\Rightarrow \boxed{K = 84\pi}$

updated. $\frac{4}{3}\pi r^3 = 84\pi t + 36\pi$

Exercise 9.4 Q20, Q21, Q22, Q23

Q.20 In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if Rs 100 Double itself in 10 years. ($\log_e 2 = 0.6931$).

Ans. Rate of change of principal $= \frac{dP}{dt}$ = ⊕ t = time years

ATQ. $\frac{dP}{dt} = r\%$ of 'P'

$\Rightarrow \frac{dP}{dt} = \frac{r}{100} \times P$ ← D. E.

$\Rightarrow \frac{dP}{P} = \left(\frac{r}{100}\right) \cdot dt$

integration

$\Rightarrow \int \frac{dP}{P} = \frac{r}{100} \cdot \int dt$

$\Rightarrow \boxed{\log(P) = \frac{r}{100} \cdot t + c}$

Time line

$t=0$	$P=100$
$t=10$	$P=200$

$\boxed{\log P = \frac{rt}{100} + c}$

$t=0, P=100$ put

$\Rightarrow \boxed{\log 100 = c}$

updated $\boxed{\log P = \frac{rt}{100} + \log 100}$

2^{nd} Cond^n. $t=10, P=200$ put

$\log m - \log n = \log \frac{m}{n}$

$\Rightarrow \log(200) = \frac{r(10)}{100} + \log 100$

$\Rightarrow \log 200 - \log 100 = \frac{r}{10}$

$\Rightarrow \log_e(2) = \frac{r}{10}$

$\Rightarrow 10(0.6931) = r \Rightarrow \boxed{r = 6.931\%}$

Q.21 In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years. ($e^{0.5} = 1.648$)

Ans. Principal = P

Rate of change of principal = $\frac{dP}{dt}$ t → time (years)

$\frac{dP}{dt} = 5\%$ of 'P'

Time line	
$t=0$	$P=1000$
$t=10$	$P=?$

$\Rightarrow \frac{dP}{dt} = \frac{\cancel{5}^{1}}{\cancel{100}_{20}} \times P$ } → DE

$\Rightarrow \frac{dP}{P} = \frac{1}{20} \cdot dt$ integrate

$\Rightarrow \int \frac{dP}{P} = \frac{1}{20} \int dt$ $\Rightarrow$

$\Rightarrow \boxed{\log(P) = \frac{1}{20}(t) + c}$

$t=0$, $P=1000$

$\Rightarrow \log(1000) = c$

$\therefore \boxed{\log P = \frac{t}{20} + \log(1000)}$

$t=10$, $P=?$

$\Rightarrow \log P = \frac{\cancel{10}}{\cancel{20}} + \log 1000$

$\Rightarrow \log P = \frac{1}{2} + \log 1000$ → exponential

$\Rightarrow \log P = \frac{1}{2} + \log 1000$

$\Rightarrow P = e^{\left(\frac{1}{2} + \log 1000\right)}$

$a^{m+n} = a^m \cdot a^n$

$\Rightarrow P = e^{1/2} \cdot e^{\log_e 1000}$

$\Rightarrow P = (1.648) \times 1000$

$\boxed{a^{\log_a b} = b}$

$\Rightarrow \boxed{P = ₹ 1648}$

Q.22 In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

rate = $\frac{dy}{dx}$

$\propto$

Time line

(i) $t=0$	$B = 1,00,000$
(ii) $t=2$	$B = 100000 + 10\%$ of 100000 $= 100000 + \frac{1\cancel{0}}{1\cancel{0}\cancel{0}} \times 10000\cancel{0}$ $= 1,10,000$
$t = t$ hours ?	$B = 2,00,000$

ATQ. Number of bacteria present = B

$$\frac{dB}{dt} \propto B$$

$$\Rightarrow \frac{dB}{dt} = K.B \leftarrow \text{D.E.}$$

Variable separation

$$\Rightarrow \frac{dB}{B} = K.dt$$

integration.

$$\Rightarrow \int \frac{dB}{B} = K \int dt$$

$$\Rightarrow \boxed{\log B = Kt + C}$$

$$\boxed{\log B = K(t) + C}$$

(i) $t=0$, $B = 100000$

$$\Rightarrow \boxed{\log 100000 = C}$$

$$\boxed{\log B = Kt + \log(100000)}$$

(ii) $t=2$, $B = 110000$

$$\log(110000) = K(2) + \log 100000$$

$$\Rightarrow \log\left(\frac{11\cancel{0000}}{1\cancel{00000}}\right) = 2K$$

$$\Rightarrow \boxed{K = \frac{1}{2}\log\left(\frac{11}{10}\right)}$$

$\log B = K.t + \log(100000)$

$K = \frac{1}{2}\log\left(\frac{11}{10}\right)$

$$\boxed{\log B = \frac{1}{2}\log\left(\frac{11}{10}\right)\cdot t + \log(100000)}$$

$t = ?$, $B = 200000$

$\Rightarrow \log(200000) = \frac{1}{2}\log\left(\frac{11}{10}\right)\cdot t + \log(100000)$

$\Rightarrow \log\left(\frac{200000}{100000}\right) = \frac{1}{2}\log\left(\frac{11}{10}\right)\times t$?

$$\Rightarrow \boxed{\frac{2\log 2}{\log\left(\frac{11}{10}\right)} = t}$$

hours.

Q. 23 The general solution of the differential equation $\boxed{\frac{dy}{dx} = e^{x+y}}$ is — (A) $e^x + e^{-y} = c$ (B) $e^x + e^y = c$ (C) $e^{-x} + e^y = c$ (D) $e^{-x} + e^{-y} = c$

$\frac{dy}{dx} = e^x \cdot e^y$

$\Rightarrow \frac{dy}{e^y} = e^x \cdot dx$

$\Rightarrow e^{-y}\cdot dy = e^x \cdot dx$

integration

$\Rightarrow \int e^{-y}\cdot dy = \int e^x \cdot dx$

$\Rightarrow \frac{e^{-y}}{-1} = e^x + c$

$\Rightarrow -e^{-y} = e^x + c$

$\Rightarrow -e^{-y} - e^x = c$

$\Rightarrow e^{-y} + e^x = -c$ $\quad -c = c_1$

$$\Rightarrow \boxed{e^{-y} + e^x = c_1}$$

Homogeneous Differential Eqrations :→

[समघातीय अवकल समीकरण] → what is it ?
→ How to solve ?

Homogeneous Functions	Homogeneous Differential Eqⁿ.

Homogeneous Functions

$f(x,y) \to$ Homogeneous Funcⁿ. of Degree n

if $\boxed{f(\lambda x, \lambda y) = \lambda^n \cdot f(x,y)}$

e.g. $f_1(x,y) = y^2 + 2xy$

$f_1(\lambda x, \lambda y) = (\lambda y)^2 + 2(\lambda x)(\lambda y)$

$= \lambda^2 y^2 + 2\lambda^2 xy$

$= \lambda^2(y^2 + 2xy) = \lambda^2 \cdot f_1(x,y)$ ✓

e.g. $f_2(x,y) = 2x + 3y + 5$

$f_2(\lambda x, \lambda y) = 2\lambda x + 3\lambda y + 5$

✗ $\neq \lambda^n \cdot f_2(x,y)$

e.g. $f_3(x,y) = \cos\left(\frac{y}{x}\right)$

$f_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right)$ ✓

$= \lambda^0 \cdot \cos\left(\frac{y}{x}\right)$

e.g. $f_4(x,y) = \sin x + \sin y$

$f_4(\lambda x, \lambda y) = \sin \lambda x + \sin \lambda y$

✗ $\neq \lambda^n \cdot f_4(x,y)$

Homogeneous Differential Eqⁿ.

$\boxed{\frac{dy}{dx} = f(x,y)} \to$ Homo. Diff. eqⁿ.

if $\boxed{f(\lambda x, \lambda y) = \lambda^0 \cdot f(x,y)}$

Homo. function of degree 0.

e.g. $(x-y)\frac{dy}{dx} = (x+2y)$

$\Rightarrow \frac{dy}{dx} = \frac{x+2y}{x-y} = f(x,y)$

$f(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \lambda^0\left(\frac{x+2y}{x-y}\right)$ ✓

e.g. $\left(1+e^{\frac{x}{y}}\right) dx = e^{\frac{x}{y}}\left(\frac{x}{y} - 1\right) \cdot dy$

$\Rightarrow \frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1+e^{x/y}} = f(x,y)$

$f(\lambda x, \lambda y) = \frac{e^{\frac{\lambda x}{\lambda y}} \cdot \left(\frac{\lambda x}{\lambda y} - 1\right)}{1 + e^{\lambda x/\lambda y}}$ ✓

$= \lambda^0 \cdot f(x,y)$

How to Solve Homogeneous Differential Equation ?

Case-I कुछ Homo. Diff. Eqⁿ मे बहुत सारी जगह $\frac{y}{x}$ बन जाता है i.e. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, तो इनमे हम $\boxed{\frac{y}{x} = v}$ मतलब $\boxed{y = vx}$ substitute कर देते है $\boxed{\text{Solve}}$ —

Case-II. कुछ Homo. Diff. Eqⁿ. मे $\frac{x}{y}$ देखने को मिलता है, i.e. $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$, तो इनमे हम $\boxed{\frac{x}{y} = v}$ मतलब $\boxed{x = v.y}$ Substitute कर देते है, $\boxed{\text{Solve}}$ —

---

e.g. Show that the differential equation

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$ is Homogeneous & Solve it.

Ans. $$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} = f(x, y)$$

$$f(\lambda x, \lambda y) = \frac{\cancel{\lambda} y \cdot \cos\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right) + \cancel{\lambda} x}{\cancel{\lambda} x \cdot \cos\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)} = \lambda^{0}\underbrace{\left(\frac{y\cos\frac{y}{x} + x}{x\cos\left(\frac{y}{x}\right)}\right)}_{f(x,y)}$$

∴ Homogeneous Diff. eqⁿ.

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} = \frac{\left(\frac{y}{x}\right)\cdot\cos\left(\frac{y}{x}\right) + 1}{\cos\left(\frac{y}{x}\right)} \quad \text{--- (1)}$$

Let $\boxed{\frac{y}{x} = v}$ $\Rightarrow y = vx$ (Substitution)

$$\frac{dy}{dx} = \frac{d(v.x)}{dx} = v.1 + x.\frac{dv}{dx}$$

$$= v + x\frac{dv}{dx}$$

Subs.

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v = \frac{\cancel{v\cos v} + 1 - \cancel{v\cos v}}{\cos v}$$

Variable separation

$$\Rightarrow \cos v.\,dv = \frac{dx}{x}$$

$$\Rightarrow \int \cos v.\,dv = \int \frac{dx}{x} \qquad \text{(integration)}$$

$$\Rightarrow \sin v = \log|x| + C$$

$$\Rightarrow \boxed{\sin\left(\frac{y}{x}\right) = \log|x| + C}$$

e.g. Show that the differential eq^n

$2y e^{x/y} \cdot dx + \left(y - 2x e^{x/y}\right) \cdot dy = 0$ is Homogeneous and find its particular solution, given that $x=0$ & $y=1$.

Ans. $2y e^{\frac{x}{y}} \cdot (dx) = \left(2x e^{\frac{x}{y}} - y\right)(dy)$

$$\Rightarrow \frac{dx}{dy} = \frac{2x e^{\frac{x}{y}} - y}{2y e^{x/y}} = \frac{\cancel{2}x \cancel{e^{x/y}}}{\cancel{2}y \cancel{e^{x/y}}} - \frac{\cancel{y}}{2\cancel{y} e^{x/y}}$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{x}{y} - \frac{1}{2e^{x/y}}} \rightarrow f(x, y)$$

$$f(\lambda x, \lambda y) = \frac{\cancel{\lambda} x}{\cancel{\lambda} y} - \frac{1}{2e^{\cancel{\lambda} x / \cancel{\lambda} y}} = \lambda^0 \left(\frac{x}{y} - \frac{1}{2e^{x/y}}\right)$$

Homo. Diff. eq^n.

$$\frac{dx}{dy} = \frac{x}{y} - \frac{1}{2e^{x/y}}$$

Substitution: $\boxed{\frac{x}{y} = v} \Rightarrow x = vy$

$$\Rightarrow \frac{dx}{dy} = \frac{d(vy)}{dy} = v + y \cdot \frac{dv}{dy}$$

$$\Rightarrow \cancel{v} + y \cdot \frac{dv}{dy} = \cancel{v} - \frac{1}{2e^v}$$

$$\Rightarrow 2e^v \cdot dv = -\frac{dy}{y}$$

Integration.

$$\Rightarrow 2\int e^v \cdot dv = -\int \frac{dy}{y}$$

$$\Rightarrow 2\int e^{v} \cdot dv = -\int \frac{dy}{y}$$

$$\Rightarrow \boxed{2\left(e^{v}\right) = -\log(y) + c}$$

$v = \frac{x}{y}$

$$\Rightarrow \boxed{2e^{x/y} = -\log(y) + c}$$ ☆ General solⁿ.

?

$x = 0,\ y = 1$ put

$$\Rightarrow 2e^{0} = -\log(1) + c$$

$$\Rightarrow 2 \times 1 = -0 + c \quad \Rightarrow \boxed{c = 2}$$

Particular solⁿ. $\boxed{2e^{x/y} = -\log(y) + 2}$

$\boxed{\text{Exercise 9.5}}$ (Homogeneous Differential Equations)

Show that the given differential equations are Homogeneous and solve each of them.

$\boxed{Q.1}$ $(x^2+xy)\,dy = (x^2+y^2)\,dx$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}} = f(x,y)$$

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+(\lambda x).(\lambda y)}$$

$$= \frac{\cancel{\lambda^2}(x^2+y^2)}{\cancel{\lambda^2}(x^2+xy)}$$

$$= \lambda^0 \left(f(x,y)\right)$$

Homogeneous Diff. eq^n.

divide by x^2 in both Nr. & Dr.

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+y^2)/x^2}{(x^2+xy)/x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)}$$

Substitution, $\boxed{\frac{y}{x} = V}$

$$\Rightarrow y = xV$$

$$\boxed{\frac{dy}{dx}} = \frac{d(xV)}{dx} = \boxed{V + x\frac{dv}{dx}}$$

Subs.

$$\Rightarrow \underbrace{V + x\frac{dv}{dx}} = \frac{1+V^2}{1+V}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+V^2}{1+V} - \frac{V}{1} \Rightarrow x\frac{dv}{dx} = \frac{1+\cancel{V^2}-V-\cancel{V^2}}{1+V}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-V}{1+V} \Rightarrow x\frac{dv}{dx} = -\left(\frac{V-1}{V+1}\right)$$

$$\Rightarrow \underbrace{\left(\frac{V+1}{V-1}\right) dv} = \underbrace{-\frac{dx}{x}}$$

integration

$$\Rightarrow \int \left(\frac{v+1}{v-1}\right)\cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{(v-1)+(2)}{(v-1)}\cdot dv = -\int \frac{dx}{x}$$

$\log m^n = n.\log m$

$$\Rightarrow \int \left(1+\frac{2}{v-1}\right)\cdot dv = -\log x + c$$

$$\Rightarrow \left[v + 2\log(v-1)\right] = -\log x + c$$

$v = \frac{y}{x}$

$$\Rightarrow \frac{y}{x} + 2\log\left(\frac{y}{x}-1\right) + \log x = c$$

$$\Rightarrow \log\left(\frac{y-x}{x}\right)^2 + \log x = -\frac{y}{x} + c$$

$$\Rightarrow \log\left(\frac{(y-x)^2}{\cancel{x^2}\,x} \times \cancel{x}\right) = \left(-\frac{y}{x} + c\right)$$

$$\Rightarrow \frac{(y-x)^2}{x} = e^{-\frac{y}{x}+c}$$

$a^{m+n} = a^m \cdot a^n$

$$\Rightarrow (y-x)^2 = x\cdot e^{-y/x}\cdot e^{c}$$

$c \rightarrow$ const

$e^c \rightarrow$ const

$\rightarrow$ New Const. $= C_1$

$$\Rightarrow \boxed{(y-x)^2 = xe^{-y/x}\cdot C_1}$$

$\boxed{Q.2}$ $y' = \dfrac{x+y}{x}$

$\Rightarrow \dfrac{dy}{dx} = \underbrace{1+\dfrac{y}{x}} = \underline{f(x,y)}$

$\Rightarrow \dfrac{dy}{dx} = 1 + \dfrac{y}{x}$

$f(\lambda x, \lambda y) = 1 + \dfrac{\cancel{\lambda} y}{\cancel{\lambda} x}$

$= 1 + \dfrac{y}{x}$

$= \lambda^{0} \cdot f(x,y)$

Homo. Diff. eq^n.

<u>Substitution</u>, $\underbrace{\dfrac{y}{x} = v} \Rightarrow y = xv$

$\Rightarrow \underline{\dfrac{dy}{dx} = \left(v + x\dfrac{dv}{dx}\right)}$

<u>Subs.</u>

$\Rightarrow \cancel{v} + x\dfrac{dv}{dx} = 1 + \cancel{v}$

$\Rightarrow dv = \dfrac{dx}{x} \Rightarrow$ Int.

$\int dv = \int \dfrac{dx}{x} \Rightarrow v = \log|x| + c$

$\Rightarrow \dfrac{y}{x} = \log|x| + c \Rightarrow \boxed{y = x\log|x| + cx}$

$\boxed{Q.3}$ $(x-y)\,dy - (x+y)\,dx = 0$

$\Rightarrow (x-y).\,dy = (x+y)\,dx$

$\Rightarrow \dfrac{dy}{dx} = \left(\dfrac{x+y}{x-y}\right) = \underline{f(x,y)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\cancel{x}\left(1+ y/x\right)}{\cancel{x}\left(1 - y/x\right)}$

$f(\lambda x, \lambda y) = \dfrac{\lambda x + \lambda y}{\lambda x - \lambda y}$

$= \lambda^{0}\left(\dfrac{x+y}{x-y}\right)$

$= \lambda^{0}\left(f(x,y)\right)$

Homo. Diff. eq^n.

<u>Substitution</u> $\boxed{y/x = v} \Rightarrow y = vx$

$$\boxed{\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$ (after substitution)

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v} = \frac{1+v^2}{1-v}$$

V.S.

$$\Rightarrow \frac{(1-v)\,dv}{(1+v^2)} = \frac{dx}{x}$$ integration

$$\Rightarrow \int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{1+v^2}\,dv - \frac{1}{2}\int \frac{2v\,dv}{(1+v^2)} = \log|x| + c$$

$2v\,dv \to dt$, $(1+v^2) \to t$

$$\Rightarrow \tan^{-1}(v) - \frac{1}{2}\log|1+v^2| = \log|x| + c$$

$v = y/x$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log\left|1+\frac{y^2}{x^2}\right| + \log|x| + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\left\{\log\left|\frac{x^2+y^2}{x^2}\right| + 2\log|x|\right\} + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\left\{\log\left|\frac{x^2+y^2}{\cancel{x^2}} \times \cancel{x^2}\right|\right\} + c$$

$$\Rightarrow \boxed{\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2) + c}$$

$\boxed{Q.4}$ $(x^2-y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow 2xy \cdot dy = (y^2-x^2)\cdot dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y^2-x^2}{2xy} = f(x,y)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\cancel{x^2}\left(\frac{y^2}{x^2}-1\right)}{\cancel{x^2}\left(2\frac{y}{x}\right)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(y/x)^2-1}{2(y/x)}$

$\downarrow \downarrow \quad \downarrow$

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{v^2-1}{2v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{v^2-1}{2v} - \dfrac{v}{1} = \dfrac{v^2-1-2v^2}{2v} = \dfrac{-v^2-1}{2v} = \dfrac{-(v^2+1)}{(2v)}$

$\Rightarrow \left(\dfrac{2v}{v^2+1}\right)\cdot dv = -\dfrac{dx}{x}$

integration

$\Rightarrow \int \dfrac{2v\,dv}{v^2+1} = -\int \dfrac{dx}{x}$ ($2v\,dv \to dt$, $v^2+1 \to t$)

$\Rightarrow \log(v^2+1) = -\log x + \log c$

$\Rightarrow \cancel{\log}\left(\dfrac{y^2}{x^2}+1\right) = \cancel{\log}\left(\dfrac{c}{x}\right)$

$f(\lambda x, \lambda y) = \dfrac{(\lambda y)^2-(\lambda x)^2}{2(\lambda x)(\lambda y)}$

$= \dfrac{\lambda^2(y^2-x^2)}{\lambda^2(2xy)}$

$= \lambda^0\,[f(x,y)]$

H. D. E

Substitution

$\dfrac{y}{x} = v \Rightarrow y = xv$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow \dfrac{y^2+x^2}{\cancel{x^2}\,x} = \dfrac{c}{\cancel{x}}$

$\Rightarrow \boxed{y^2+x^2 = cx}$

$\boxed{Q.5}$ $x^2 \cdot \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)}_{\to f(x,y)}$$

$f(\lambda x, \lambda y)$

$$= 1 - 2\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)^2 + \left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)$$

$$= \lambda^0 \left(f(x,y) \right)$$

H.D.E

Substitution. $\dfrac{y}{x} = v \Rightarrow y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = 1 - 2v^2 + \cancel{v}$$

$$\Rightarrow \frac{dv}{1-2v^2} = \frac{dx}{x}$$

integration

$$\Rightarrow \int \frac{dv}{1-(\sqrt{2}v)^2} = \int \frac{dx}{x}$$

$v = \dfrac{y}{x}$

$$\Rightarrow \frac{\frac{1}{2\times 1} \cdot \log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|}{\sqrt{2}} = \log|x| + c$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log\left|\frac{1+\sqrt{2}\cdot\frac{y}{x}}{1-\sqrt{2}\cdot y/x}\right| = \log|x| + c$$

$$\Rightarrow \boxed{\frac{1}{2\sqrt{2}} \log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + c}$$

$\boxed{\text{Exercise 9.5}}$ (Homogeneous Diff. eqⁿ.)

Show that the given diff. eqⁿ. are Homogeneous & solve each of them.

Q.6 $\quad \dfrac{x\,dy - y\,dx = \sqrt{x^2+y^2}\cdot dx}{dx}$

$\Rightarrow x\left(\dfrac{dy}{dx}\right) - y = \sqrt{x^2+y^2}$

$\Rightarrow x\cdot\dfrac{dy}{dx} = \sqrt{x^2+y^2} + y$

$\Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+y^2}+y}{x}} = f(x,y)$

$f(\lambda x, \lambda y)$

$= \dfrac{\sqrt{(\lambda x)^2+(\lambda y)^2}+\lambda y}{\lambda x}$

$= \dfrac{\cancel{\lambda}\left(\sqrt{x^2+y^2}+y\right)}{\cancel{\lambda} x}$

$= \lambda^0\left(f(x,y)\right)$

Homo. D.E.

$\Rightarrow \left(\dfrac{dy}{dx}\right) = \dfrac{\cancel{x}\left\{\sqrt{1+\left(\frac{y}{x}\right)^2}+\left(\frac{y}{x}\right)\right\}}{\cancel{x}}$

Substitution, $\dfrac{y}{x} = v \Rightarrow \boxed{y = xv}$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow \cancel{v} + x\dfrac{dv}{dx} = \sqrt{1+v^2} + \cancel{v}$

$\Rightarrow \dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$ (Integration)

$\Rightarrow \displaystyle\int \dfrac{dv}{\sqrt{1+v^2}} = \int \dfrac{dx}{x} \Rightarrow \boxed{\log\left(v+\sqrt{1+v^2}\right) = \log x + \log c}$

$$\Rightarrow \log\left(v+\sqrt{1+v^2}\right) = \log x + \log c$$

put $v = y/x$

$$\Rightarrow \cancel{\log}\left(\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right) = \cancel{\log}(xc)$$

$$\Rightarrow \frac{y}{x}+\sqrt{\frac{x^2+y^2}{x^2}} = xc \Rightarrow \boxed{\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x} = xc}$$

$$\Rightarrow \boxed{y+\sqrt{x^2+y^2} = x^2 . c}$$

Q.7 $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y . dx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy$

$$\Rightarrow \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}\cdot y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}\cdot x} = \frac{dy}{dx}$$

$$\Rightarrow \boxed{\underbrace{\frac{\frac{y}{x}\left\{\cos\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)\cdot\sin\left(\frac{y}{x}\right)\right\}}{\left\{\frac{y}{x}\cdot\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)\right\}}}_{f(x,y)} = \frac{dy}{dx}}$$

H. D. E.

$$\boxed{f(\lambda x, \lambda y) = \lambda^0 . f(x, y)}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cdot\left\{\cos\frac{y}{x} + \frac{y}{x}\cdot\sin\frac{y}{x}\right\}}{\frac{y}{x}\cdot\sin\frac{y}{x} - \cos\frac{y}{x}}$$

Substitution, $\underbrace{\frac{y}{x}} = v \Rightarrow y = v.x$

$$\frac{dy}{dx} = \underbrace{v + x\frac{dv}{dx}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v.(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + \cancel{v^2\sin v} - \cancel{v^2\sin v} + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

(Variable separation)

$$\Rightarrow \left(\frac{v\sin v - \cos v}{v\cos v}\right).dv = 2.\frac{dx}{x}$$

Integrate

$$\Rightarrow \int\left(\tan v - \frac{1}{v}\right).dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log(c)$$

$$\Rightarrow \cancel{\log}\left|\frac{\sec v}{v}\right| = \cancel{\log}\left|x^2.c\right|$$

$v = y/x$

$$\Rightarrow \frac{\sec v}{v} = x^2 \cdot c$$

$$v = y/x$$

$$\Rightarrow \frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)} = x^2 \cdot c$$

$$\Rightarrow \boxed{\sec(y/x) = xy \cdot c}$$

$$\frac{1}{c} = xy \cdot \cos\left(\frac{y}{x}\right)$$

$$\boxed{c_1 = xy \cdot \cos\left(\frac{y}{x}\right)}$$

Q.8 $x\left(\frac{dy}{dx}\right) - y + x \sin\left(\frac{y}{x}\right) = 0$

$$\Rightarrow x \cdot \frac{dy}{dx} = y - x\sin\left(y/x\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)} \to f(x,y)$$

$$f(\lambda x, \lambda y)$$

$$= \frac{\cancel{\lambda} y}{\cancel{\lambda} x} - \sin\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)$$

$$= \frac{y}{x} - \sin\left(\frac{y}{x}\right)$$

$$= \lambda^0 \cdot f(x,y)$$

H.D.E

Substitution :→

$$\frac{y}{x} = v \Rightarrow y = x.v$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} - \sin v$$

$$\Rightarrow \frac{dv}{\sin v} = -\frac{dx}{x}$$ integration

$$\Rightarrow \int \frac{dv}{\sin v} \Rightarrow \int \operatorname{cosec} v \cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log|\operatorname{cosec} v - \cot v| = -\log x + \log c$$

$$\Rightarrow \cancel{\log}\left|\frac{1}{\sin v} - \frac{\cos v}{\sin v}\right| = \cancel{\log}\left|\frac{c}{x}\right|$$

$$\Rightarrow \frac{(1-\cos v)}{(\sin v)} = \frac{c}{x}$$

$v = y/x$

$$\Rightarrow \boxed{x\left(1-\cos\frac{y}{x}\right) = c.\sin\left(\frac{y}{x}\right)}$$

$\boxed{Q.9}$ $y\,dx + x\log\left(\frac{y}{x}\right).dy - 2x\,dy = 0$

dy

$$\Rightarrow y.\left(\frac{dx}{dy}\right) + x\log\left(\frac{x}{y}\right)^{-1} - 2x = 0$$

$$\Rightarrow y.\left(\frac{dx}{dy}\right) - x\log\left(\frac{x}{y}\right) - 2x = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{x.\log\left(\frac{x}{y}\right) + 2x}{y}$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \underbrace{\frac{x}{y}.\log\left(\frac{x}{y}\right) + 2\left(\frac{x}{y}\right)}_{f(x,y)}}$$

$f(\lambda x, \lambda y)$

$= \lambda^{0} f(x,y)$

H.D.E

Substitution. $\frac{x}{y} = v \Rightarrow x = yv$

$$\Rightarrow \frac{dx}{dy} = v + y.\frac{dv}{dy}$$

$$\Rightarrow v + y.\frac{dv}{dy} = v.\log(v) + 2v$$

$$\Rightarrow y \cdot \frac{dv}{dy} = \underbrace{v \cdot \log v + v} = v(\log v + 1)$$

$$\Rightarrow \frac{dv}{v \cdot (\log v + 1)} = \frac{dy}{y}$$

Integration

$$\Rightarrow \int \frac{dv}{v \cdot (\log v + 1)} = \int \frac{dy}{y}$$

Legend

$dv \to dt$, $(\log v + 1) \to t$

$$\Rightarrow \log|\log v + 1| = \log|y| + \log C$$

$$\Rightarrow \cancel{\log}|\log v + 1| = \cancel{\log}(yC)$$

$v \to x/y$

$$V = \frac{x}{y}$$

$\Rightarrow$ ~~log~~

$$\Rightarrow \boxed{\log\left(\frac{x}{y}\right) + 1 = yC}$$

Q.10 $\left(1 + e^{x/y}\right) \cdot dx + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot dy = 0$

$$\Rightarrow \left(1 + e^{x/y}\right) \cdot dx = e^{x/y} \cdot \left(\frac{x}{y} - 1\right) \cdot dy$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{\left(1 + e^{x/y}\right)}} = f(x, y)$$

$$f(\lambda x, \lambda y) = \lambda^0 \cdot f(x, y)$$

H.D.E

$$\Rightarrow \frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{1+e^{x/y}}$$

$\downarrow$ After subs.

Subst.

$$\frac{x}{y} = v$$

$$\Rightarrow x = y.v$$

$$\Rightarrow \frac{dx}{dy} = v + y.\frac{dv}{dy}$$

$$\Rightarrow v + y.\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$$

$$\Rightarrow y.\frac{dv}{dy} = \frac{e^v.v - e^v}{1+e^v} - v = \frac{\cancel{v.e^v} - e^v - v - \cancel{v e^v}}{1+e^v}$$

$$\Rightarrow y.\frac{dv}{dy} = \frac{-(e^v+v)}{(e^v+1)} \Rightarrow \left(\frac{e^v+1}{e^v+v}\right)dv = -\frac{dy}{y}$$

integration,

$$\Rightarrow \int \frac{e^v+1}{e^v+v}\,dv = -\int\frac{dy}{y} \qquad v = \frac{x}{y}$$

($e^v+1\,dv \to dt$, $e^v+v \to t$)

$$\Rightarrow \log(e^v+v) = -\log y + \log c$$

$$\Rightarrow \cancel{\log}\left(e^{x/y} + \frac{x}{y}\right) = \cancel{\log}\left(\frac{c}{y}\right)$$

$$\Rightarrow \left(e^{x/y} + \frac{x}{y}\right) = \frac{c}{y}$$

$$\Rightarrow \boxed{y.e^{x/y} + x = c}$$

$\boxed{\text{Exercise 9.5}}$ (Homogeneous Differential Equations)

Find the particular solution satisfying the given condition:

$\boxed{Q.11}$ $(x+y).dy + (x-y).dx = 0$; $y=1$ when $x=1$.

$$\Rightarrow (x+y)dy = (y-x).dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x} = \frac{\cancel{x}(y/x - 1)}{\cancel{x}(y/x + 1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y/x - 1}{y/x + 1}$$

Substitution, $y/x = v$

$$\Rightarrow y = xv$$

$$\Rightarrow \frac{dy}{dx} = \frac{d(xv)}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - \frac{v}{1} = \frac{\cancel{v}-1-v^2-\cancel{v}}{v+1} = -\frac{(v^2+1)}{(v+1)}$$

variable separation

$$\Rightarrow \left(\frac{v+1}{v^2+1}\right)dv = -\frac{dx}{x}$$

integration,

$$\Rightarrow \int\left(\frac{v}{v^2+1} + \frac{1}{v^2+1}\right).dv = -\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\frac{2v.dv \to dt}{v^2+1 \to t} + \int\frac{1}{v^2+1}.dv = -\log|x| + c$$

$$\Rightarrow \frac{1}{2}\log|v^2+1| + \tan^{-1}v = -\log|x| + c$$

$v = y/x$

$$\Rightarrow \left(\frac{1}{2}\log(v^2+1) + \tan^{-1} v = -\log|x| + c\right) \times 2$$

$v = y/x$

$$\Rightarrow \log\left(\frac{y^2}{x^2}+1\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = -2\log(x) + 2c$$

$$\Rightarrow \underbrace{\log\left(\frac{y^2+x^2}{x^2}\right) + \log(x^2)} + 2\tan^{-1}\left(\frac{y}{x}\right) = 2c$$

$$\Rightarrow \log\left(\frac{y^2+x^2}{\cancel{x^2}} \times \cancel{x^2}\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = 2c$$

$$\Rightarrow \boxed{\log(y^2+x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = 2c}$$

?

$\underbrace{y=1, x=1}$ put

$$\Rightarrow \log(2) + 2\underbrace{\tan^{-1}(1)} = 2c$$

$\tan^{-1}(1) = \frac{\pi}{4}$

$$\Rightarrow \log 2 + \cancel{2} \times \frac{\pi}{\cancel{4}_2} = 2c$$

$$\Rightarrow \underbrace{\log 2 + \frac{\pi}{2}} = 2c$$

Particular Solution.

$$\log(y^2+x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \log 2 + \frac{\pi}{2}$$

Q.12 $x^2\,dy + (xy+y^2)\,dx = 0$; $y=1$ when $x=1$

$$\Rightarrow x^2\,dy = -(xy+y^2)\,dx$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{xy+y^2}{x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{y}{x} + \left(\frac{y}{x}\right)^2\right]$$

Substitution: $\frac{y}{x} = v$

$$\Rightarrow y = xv$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitution:

$$v + x\frac{dv}{dx} = -(v+v^2).$$

$$\Rightarrow x\frac{dv}{dx} = -(2v+v^2)$$ V.S.

$$\Rightarrow \frac{dv}{2v+v^2} = -\frac{dx}{x}$$ integration

$$\Rightarrow \int \frac{dv}{v(v+2)} = -\int \frac{dx}{x}$$ $\frac{A}{V} + \frac{B}{V+2}$

By Partial Fraction

$$\Rightarrow \int \left(\frac{1/2}{v} - \frac{1/2}{v+2}\right).dv = -\log|x| + \log c$$

$$\Rightarrow \frac{1}{2}\log|v| - \frac{1}{2}\log|v+2| = -\log|x| + \log c$$

$$\Rightarrow \frac{1}{2}\left\{\log\left|\frac{v}{v+2}\right|\right\} = \log\left|\frac{c}{x}\right|$$

$V = \frac{y}{x}$

$$\Rightarrow \log\left|\frac{v}{v+2}\right| = 2\log\left|\frac{c}{x}\right|$$

$n.\log m = \log m^n$

$$\Rightarrow \cancel{\log}\left|\frac{y/x}{y/x+2}\right| = \cancel{\log}\left(\frac{c^2}{x^2}\right)$$

$$\Rightarrow \frac{(y/\cancel{x})}{\left(\frac{y+2x}{\cancel{x}}\right)} = \frac{c^2}{x^2} \Rightarrow \frac{y}{y+2x} = \frac{c^2}{x^2}$$

$$\Rightarrow \boxed{yx^2 = (y+2x).c^2}$$ $x=1, y=1$

$$\Rightarrow 1 = 3.c^2$$

$$\Rightarrow c^2 = \frac{1}{3}$$

$$yx^2 = (y+2x).\frac{1}{3}$$

$$\Rightarrow \boxed{3yx^2 = (y+2x)}$$

Q.13 $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$; $y = \frac{\pi}{4}$ when $x=1$

$\Rightarrow x\,dy = \left(y - x\cdot\sin^2\frac{y}{x}\right)\cdot dx$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$

Substitution.

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} - \sin^2(v)$

$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{x}$ integration

$\Rightarrow \int\frac{dv}{\sin^2 v} = -\int\frac{dx}{x} \Rightarrow \int\operatorname{cosec}^2 v\cdot dv = -\int\frac{dx}{x}$

$\Rightarrow -\cot v = -\log|x| + c$ $\quad v = \frac{y}{x}$

$\Rightarrow \boxed{-\cot\left(\frac{y}{x}\right) = -\log|x| + c}$

$x=1,\ y=\frac{\pi}{4}$

$-\cot\left(\frac{\pi}{4}\right) = -\log|1| + c$

$1 \qquad 0$

$\Rightarrow \boxed{-1 = c}$

$\Rightarrow \cancel{-}\cot\left(\frac{y}{x}\right) = \cancel{-}\log|x| \cancel{-} 1$

$+ \qquad +$

$\Rightarrow \boxed{\cot\left(\frac{y}{x}\right) = \log_e|x| + 1}$

$1 = \log_e e$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log_e|x| + \log_e e$

$\log m + \log n = \log mn$

$\Rightarrow \boxed{\cot\left(\frac{y}{x}\right) = \log|ex|}$

Q.14 $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$; $y = 0$ when $x = 1$.

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right)$

Substitution.

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} - \text{cosec}(v)$

$\Rightarrow \frac{dv}{\text{cosec}\,v} = -\frac{dx}{x}$

integration

$\Rightarrow \int \sin v . dv = -\int \frac{dx}{x} \Rightarrow -\cos v = -\log|x| - c$

$\Rightarrow \boxed{\cos v = \log|x| + c}$ $v = \frac{y}{x}$

$\Rightarrow \boxed{\cos\left(\frac{y}{x}\right) = \log|x| + c}$ $y = 0,\ x = 1$

$\cos 0 = \cancel{\log|1|} + c$

$\Rightarrow \boxed{1 = c}$

$\Rightarrow \boxed{\cos\left(\frac{y}{x}\right) = \log|x| + 1}$

$\Rightarrow \cos\frac{y}{x} = \log|x| + \log e$

$\Rightarrow \boxed{\cos\frac{y}{x} = \log|xe|}$

$\boxed{Q.15}$ $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$; $y = 2$ when $x = 1$

$\Rightarrow 2xy + y^2 = 2x^2 \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{2x}y}{\cancel{2x}x} + \frac{y^2}{2x^2} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2} \rightarrow f\left(\frac{y}{x}\right)$

Substitution,

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \frac{1}{2}v^2$

$\Rightarrow \frac{dv}{v^2} = \frac{1}{2}\frac{dx}{x}$ integrate

$\Rightarrow \int \frac{dv}{v^2} = \frac{1}{2}\int \frac{dx}{x} \Rightarrow \int v^{-2} \cdot dv = \frac{1}{2}\log|x| + c$

$\Rightarrow -\frac{1}{v} = \frac{1}{2}\log|x| + c \Rightarrow \boxed{-\frac{x}{y} = \frac{1}{2}\log|x| + c}$

$x = 1,\ y = 2$

$-\frac{1}{2} = \frac{1}{2}\cancel{\log|1|} + c$

$\Rightarrow c = -\frac{1}{2}$

$\Rightarrow -\frac{x}{y} = \frac{1}{2}\log|x| - \frac{1}{2}$

$\Rightarrow -\frac{x}{y} = \frac{\log|x| - 1}{2}$

$\Rightarrow \frac{-2x}{\log|x| - 1} = y$

$\Rightarrow \boxed{y = \frac{2x}{1 - \log|x|}}$

Q.16 & Q.17 Solved in Video Directly.

Linear Differential Equations [रैखिक अवकल समीकरण]

Pattern - 1

$$\frac{dy}{dx} + P.y = Q$$

Constant or functions of x

e.g. $\frac{dy}{dx} + 1.y = \sin x$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{1}{x}$$

Steps to Solve $\boxed{\frac{dy}{dx} + P.y = Q}$ (x)

(i) Find the Integrating Factor

$$I.F. = e^{\int P.dx}$$

(समाकलन गुणक)

(ii) Solution :

$$\boxed{y(I.F.) = \int (Q \times I.F.).dx + c}$$

Pattern - 2

$$\frac{dx}{dy} + P.x = Q$$

Constant or functions of y

e.g. $\frac{dx}{dy} - \frac{x}{y} = 2y$

$$\frac{dx}{dy} + x\cot y = 2y + y^2\cot y$$

Steps to Solve $\frac{dx}{dy} + P.x = Q$ (y)

(i) Find the Integrating Factor

$$I.F. = e^{\int P.dy}$$

(ii) Solution

$$x(I.F.) = \int Q(I.F.).dy + c$$

e.g. Find the particular solution of the differential equation $\frac{dy}{dx} + y.\cot x = 2x + x^2.\cot x$ $(x \neq 0)$

given that $y=0$ when $x = \frac{\pi}{2}$.

$\frac{dy}{dx} + P.y = Q$

Ans. $\frac{dy}{dx} + y \underbrace{\cot x}_{P} = \underbrace{2x + x^2.\cot x}_{Q}$

Linear Diff. Eq^n.

(LDE)

Integrating factor

$\boxed{I.F. = e^{\int P.dx}} \Rightarrow I.F. = e^{\int \cot x . dx}$

$\Rightarrow I.F. = e^{\log \sin x} = \sin x$

Solution, $\boxed{y(I.F) = \int Q.(I.F). dx + C}$

$\Rightarrow y.(\sin x) = \int (2x + x^2.\cot x).\sin x.dx + C$

$\Rightarrow y.\sin x = \int \left(2x\sin x + x^2.\frac{\cos x}{\cancel{\sin x}}.\cancel{\sin x}\right).dx + C$

$\Rightarrow y\sin x = \int (2x.\sin x).dx + \int \overset{I}{x^2}.\overset{II}{\cos x}.dx + C$

ILATE

Int. by Parts.

$\Rightarrow y\sin x = \cancel{\int 2x\sin x .dx} + \left(x^2.\sin x - \cancel{\int 2x.\sin x.dx}\right) + C$

$\boxed{\int I.II = I.\int II - \int (I'.\int II)}$ $\Rightarrow \boxed{y\sin x = x^2\sin x + C}$

Solution. $\boxed{y \sin x = x^2 \sin x + C}$ ← General Solⁿ;

$y = 0,\ x = \frac{\pi}{2} \rightarrow 0 = \frac{\pi^2}{4} \cdot \sin\frac{\pi}{2} + C$

$\sin\frac{\pi}{2} = 1$

Particular solⁿ;

$\Rightarrow \boxed{C = -\frac{\pi^2}{4}}$

$$\boxed{y \sin x = x^2 \sin x - \frac{\pi^2}{4}}$$

$$\Rightarrow \boxed{y = x^2 - \frac{\pi^2}{4 \sin x}} \checkmark$$

[redacted] **e.g.** Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

$$\boxed{\frac{dx}{dy} + P \cdot x = Q}$$

$\Rightarrow y \cdot dx = (x + 2y^2) \cdot dy$

$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$

$\Rightarrow \boxed{\frac{dx}{dy} - \frac{x}{y} = 2y}$ $P = -\frac{1}{y}$ ✓ $Q = 2y$ ✓

$I.F = e^{\int P \cdot dy}$

$= e^{\int -\frac{1}{y} \cdot dy}$

$= e^{-\log(y)} = \cancel{e}^{\log(y^{-1})}$

$= y^{-1} = \frac{1}{y} = I.F$

Solution. $\boxed{x(I.F) = \int Q \cdot (I.F)\, dy + C}$

$\Rightarrow x\left(\frac{1}{y}\right) = \int (2\cancel{y}) \cdot \frac{1}{\cancel{y}} \cdot dy + C$

$\Rightarrow \frac{x}{y} = 2\int dy + C$

$\Rightarrow \boxed{\frac{x}{y} = 2y + C} \Rightarrow \boxed{x = 2y^2 + Cy} \checkmark$

$\boxed{\text{Exercise 9.6}}$ (Linear Differential Equations)

For each of the differential equations given, find the general solution —

$\boxed{Q.1}$ $\frac{dy}{dx} + 2y = \sin x$

$$\boxed{\frac{dy}{dx} + P.y = Q} \quad \text{L.D.E}$$

Here $P = 2$

$Q = \sin x$

Integrating Factor

$$I.F = e^{\int P.dx}$$

$$IF = e^{\int 2.dx} = e^{2x}$$

Solution:

$$\boxed{y(I.F) = \int [Q \times (I.F)].dx + c}$$

$$\Rightarrow y.e^{2x} = \underbrace{\int \sin x . e^{2x} . dx}_{} + c \quad \text{—— (1)}$$

Integration by Parts. ILATE

$$I = \int \overset{I}{\sin x} . \overset{II}{e^{2x}} . dx$$

ILATE

$$\boxed{\int I . II = I . \int II - \int (I' . \int II) . dx}$$

$$I = \sin x . \frac{e^{2x}}{2} - \int \overset{I}{\cos x} . \overset{II}{\frac{e^{2x}}{2}} . dx$$

ILATE

$$I = \frac{e^{2x} . \sin x}{2} - \left[\cos x . \frac{e^{2x}}{4} + \int \sin x . \frac{e^{2x}}{4} . dx\right]$$

$$I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}\int \sin x . e^{2x} . dx$$

I

$$\Rightarrow I = \frac{e^{2x}}{4}\left\{2\sin x - \cos x\right\} - \frac{I}{4}$$

$$\Rightarrow \frac{I}{1} + \frac{I}{4} = \frac{e^{2x}}{4}\left\{2\sin x - \cos x\right\}$$

$$\Rightarrow \frac{5I}{\cancel{4}} = \frac{e^{2x}}{\cancel{4}}\left\{2\sin x - \cos x\right\}$$

$$\Rightarrow \boxed{I = \frac{e^{2x}}{5}\left(2\sin x - \cos x\right)} \quad \text{--- (2)}$$

By eq^n (1) & (2) $\Rightarrow$ $y.e^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$

$$\Rightarrow y = \frac{\frac{e^{2x}}{5}(2\sin x - \cos x) + C}{e^{2x}} = \frac{1}{5}(2\sin x - \cos x) + C.e^{-2x}$$

Q.2 $\frac{dy}{dx} + 3y = e^{-2x}$ L.D.E $\boxed{\frac{dy}{dx} + P.y = Q}$ x / Constant

$\boxed{P = 3, \quad Q = e^{-2x}}$

$$I.F. = e^{\int P.dx} = e^{\int 3.dx} = e^{3x}$$

Solution, $\boxed{y(I.F) = \int Q(I.F).dx + c}$

$$\Rightarrow y \times e^{3x} = \int e^{-2x}.e^{3x}.dx + c$$

$$\Rightarrow y.e^{3x} = \int e^{x}.dx + c$$

$$\Rightarrow \boxed{y.e^{3x} = e^{x} + c} \Rightarrow \boxed{y = e^{-2x} + c.e^{-3x}}$$

$\boxed{Q.3}$ $\boxed{\frac{dy}{dx} + \frac{y}{x} = x^2}$ L.D.E $\frac{dy}{dx} + P.y = Q$

$P = \frac{1}{x}$, $Q = x^2$ $\quad$ $x \quad x$ $\quad$ $a^{\log_a x} = x$

$$I.F. = e^{\int P.dx} = e^{\int \frac{1}{x}.dx} = e^{\log_e x} = x$$

Solution: $\boxed{y.(I.F) = \int Q.(I.F).dx + C}$

$$\Rightarrow y.(x) = \int x^2.x.dx + C$$

$$\Rightarrow xy = \int x^3.dx + C \Rightarrow \boxed{xy = \frac{x^4}{4} + C}$$

$\boxed{Q.4}$ $\frac{dy}{dx} + (\sec x)y = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$

$\boxed{\frac{dy}{dx} + P.y = Q}$ L.D.E.

$$I.F. = e^{\int P.dx}$$
$$= e^{\int \sec x.dx}$$
$$= e^{\log[\sec x + \tan x]}$$
$$= (\sec x + \tan x)$$

Solution:

$$y.(I.F) = \int Q.(I.F).dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x.(\sec x + \tan x).dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int (\sec x \tan x + \tan^2 x).dx + C$$

$\tan^2\theta = \sec^2\theta - 1$

$$\Rightarrow y(\sec x + \tan x) = \int (\sec x.\tan x + \sec^2 x - 1).dx + C$$

$$\Rightarrow y(\sec x + \tan x) = (\sec x + \tan x - x) + C$$

$\boxed{Q.5}$ $\cos^2 x \cdot \frac{dy}{dx} + y = \tan x$ $\left(0 \le x < \frac{\pi}{2}\right)$

$\boxed{L.D.E}$

$\cos^2 x$

Divide

$\boxed{\frac{dy}{dx} + P.y = Q}$

$\Rightarrow \boxed{\frac{dy}{dx} + y.\sec^2 x = \tan x.\sec^2 x}$ //

(x) of terms.

$P = \sec^2 x$

$Q = \tan x.\sec^2 x$

$$\boxed{\begin{aligned} I.F. &= e^{\int P.dx} \\ I.F &= e^{\int \sec^2 x.dx} \\ I.F. &= e^{\tan x} \end{aligned}}$$

Solution.

$\boxed{y(I.F) = \int Q.(I.F).dx + C}$

$\Rightarrow y.(e^{\tan x}) = \int \tan x.\sec^2 x.\, e^{\tan x}.dx + C$

$\Rightarrow y.e^{\tan x} = \int e^{\tan x}.\tan x.\sec^2 x.dx + c$

By substitution. $\tan x = t$

$\sec^2 x.dx = dt$

$\Rightarrow y.e^{\tan x} = \int \overset{II}{e^t}.\overset{I}{t}.dt + c$

ILATE

(Integration by Parts)

$\boxed{\int I.II = I.\int II - \int [I'.\int II]}$

$\Rightarrow y.e^{\tan x} = t.e^t - \int (1.e^t).dt + c$

$\Rightarrow y.e^{\tan x} = (t.e^t - e^t) + c$

$t = \tan x$

$\Rightarrow \boxed{y.e^{\tan x} = e^{\tan x}(\tan x - 1) + c}$

Exercise 9.6 (Linear Differential Equation)

Q.6 $x\frac{dy}{dx} + 2y = x^2 \cdot \log x$ ← Find the general Solution

x

L.D.E.

$\frac{dy}{dx} + P \cdot y = Q$

$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x\log x$

$P = \frac{2}{x}$ $Q = x\log x$

Integrating Factor $I.F. = e^{\int P \cdot dx} = e^{\int \frac{2}{x} \cdot dx}$

$\Rightarrow I.F. = e^{2\log x} = \cancel{e}^{\log x^2} = x^2$

Solution: $\boxed{y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C}$

$\Rightarrow y \cdot x^2 = \int x\log x \cdot x^2 \cdot dx + C$

$\Rightarrow y \cdot x^2 = \int \overset{II}{x^3} \cdot \overset{I}{\log x} \cdot dx + c$

ILATE

Integration by Parts

$\boxed{\int I \cdot II = I \int II - \int \left(I' \cdot \int II\right)}$

$\Rightarrow yx^2 = \log x \cdot \frac{x^4}{4} - \int \left(\frac{1}{x} \cdot \frac{x^4}{4}\right) \cdot dx + C$

$\Rightarrow yx^2 = \frac{\log x \cdot x^4}{4} - \frac{1}{4}\int x^3 \cdot dx + C$

$\Rightarrow yx^2 = \frac{x^4 \cdot \log x}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right) + C$

$$\Rightarrow y x^2 = \frac{x^4}{16}\{4\log x - 1\} + c$$

$$\Rightarrow \boxed{y = \frac{x^2}{16}\{4\log x - 1\} + c.x^{-2}}$$

Q.7 $\quad x\log x.\frac{dy}{dx} + y = \frac{2}{x}\log x$

L.D.E

$x\log x$

$$\boxed{\frac{dy}{dx} + P.y = Q}$$

$$\Rightarrow \boxed{\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}}$$

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x^2}$$

$$\underline{I.F} = e^{\int P.dx} = e^{\int \frac{1}{x\log x}.dx}$$

$$= e^{\log(\log x)}$$

$$= (\log x)$$

$\int \frac{1}{x\log x}.dx$

$\log x = t$

$\frac{1}{x}.dx = dt$

$\int \frac{dt}{t} = \log(t) + c$

$= \log(\log x)$

Solution: $\boxed{y.(IF) = \int Q.(IF).dx + c}$

$$\Rightarrow y.\log x = \int \overset{II}{\frac{2}{x^2}}.\overset{I}{\log x}.dx + c$$

ILATE

$$\Rightarrow y.\log x = \log x.\int \frac{2}{x^2}.dx - \int\left[\left(\frac{1}{x}\right).\int \frac{2}{x^2}.dx\right].dx + c$$

$$\Rightarrow y\log x = \log x.\left(-\frac{2}{x}\right) - \int \frac{1}{x}.\left(-\frac{2}{x}\right).dx + c$$

$$\Rightarrow y\log x = \frac{-2\log x}{x} + \int \frac{2}{x^2}.dx + c$$

$-2/x$

$$\Rightarrow y \log x = \frac{-2\log x}{x} + \int \frac{2}{x^2}\,dx + c \rightarrow -\frac{2}{x}$$

$$\Rightarrow y \log x = \underbrace{-\frac{2\log x}{x} - \frac{2}{x}} + c$$

$$\Rightarrow \boxed{y \log x = -\frac{2}{x}\left\{\log x + 1\right\} + c}$$

Q.8 $(1+x^2)\,dy + 2xy\,dx = \cot x\,dx \quad (x \neq 0)$

$\div\ dx$

L.D.E

$$\boxed{\frac{dy}{dx} + P y = Q}$$

x

$$\Rightarrow \frac{(1+x^2)\frac{dy}{dx} + 2xy = \cot x}{(1+x^2)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{\cot x}{1+x^2}} \equiv \boxed{\frac{dy}{dx} + Py = Q}$$

P, Q

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}\,dx} = e^{\log(1+x^2)} = (1+x^2)$$

Solution: $\boxed{y\cdot(IF) = \int Q(IF)\,dx + c}$

$$\Rightarrow y\cdot(1+x^2) = \int \frac{\cot x}{\cancel{(1+x^2)}}\cdot\cancel{(1+x^2)}\,dx + c$$

$$\Rightarrow y\cdot(1+x^2) = \log|\sin x| + c$$

$$\Rightarrow \boxed{y = \frac{\log|\sin x|}{1+x^2} + \frac{c}{1+x^2}} = (1+x^2)^{-1}\cdot\log|\sin x| + c\cdot(1+x^2)^{-1}$$

Q.9 $x \cdot \frac{dy}{dx} + y - x + xy\cot x = 0$, $(x \neq 0)$

$\div x$

$$\boxed{\frac{dy}{dx} + P \cdot y = Q}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} - 1 + y\cot x = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} + y\left(\frac{1}{x} + \cot x\right) = 1}$$

P Q

$\log m + \log n = \log mn$

$$I.F. = e^{\int P \cdot dx} = e^{\int\left(\frac{1}{x} + \cot x\right) \cdot dx} = e^{\log x + \log \sin x}$$

$$IF = e^{\log_e(x \sin x)} = x \sin x$$

Solution: $\boxed{y \cdot (IF) = \int Q \cdot (IF) \cdot dx + C}$

$$\Rightarrow y \cdot (x \sin x) = \int 1 \cdot \overset{I}{x} \overset{II}{\sin x} \cdot dx + C$$

ILATE

Int. by Parts.

$$\Rightarrow y \cdot (x \sin x) = x \int \sin x \cdot dx - \int \left(1 \cdot (-\cos x)\right) \cdot dx + C$$

$$\Rightarrow y \cdot (x \sin x) = -x\cos x + \sin x + C$$

$$\Rightarrow y = -\frac{\cancel{x}\cos x}{\cancel{x}\sin x} + \frac{\cancel{\sin x}}{x\cancel{\sin x}} + \frac{C}{x \sin x}$$

$$\Rightarrow \boxed{y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}}$$

$\boxed{\text{Exercise 9.6}}$ (Linear Differential Equations)

Find the general Solution

$\boxed{Q10}$ $(x+y)\cdot\frac{dy}{dx} = 1 \Rightarrow \underbrace{\frac{dy}{dx} = \frac{1}{x+y}} \Rightarrow \frac{dx}{dy} = x+y$

$$\Rightarrow \boxed{\frac{dx}{dy} - x = y} \qquad \boxed{\frac{dx}{dy} + P\cdot x = Q}$$

Standard L.D.E.

$\left.\begin{matrix} P = -1 \\ Q = y \end{matrix}\right\}$ I.F. $= e^{\int P\cdot dy} = e^{-\int 1\cdot dy}$

(Integrating Factor) $= e^{-y}$

Solution: $\boxed{x\,(I.F) = \int Q\,(I.F)\cdot dy + c}$

$$\Rightarrow x\cdot e^{-y} = \int \overset{I}{y}\cdot \overset{II}{e^{-y}}\cdot dy + c \qquad \int II = \int e^{-y} = \frac{e^{-y}}{-1}$$

ILATE Int. by Parts

$$\boxed{\int I\cdot II = I\int II - \int\left[I'\cdot\int II\right]}$$

$$\Rightarrow x e^{-y} = \underbrace{y\,(-e^{-y}) - \int\left[1\cdot(-e^{-y})\right]\cdot dy} + c$$

$$\Rightarrow x e^{-y} = -y\cdot e^{-y} + (-e^{-y}) + c$$

$$\Rightarrow x e^{-y} = -y e^{-y} - e^{-y} + c$$

$$\Rightarrow \boxed{x e^{-y} + y e^{-y} + e^{-y} = c}$$

$$\Rightarrow \boxed{(x + y + 1) = c\cdot e^{y}}$$

Q.11 $y\,dx + (x - y^2)\,dy = 0$

$$\overline{\quad y \cdot dy \quad}$$

L.D.E.

$$\boxed{\frac{dx}{dy} + P \cdot x = Q}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{x - y^2}{y}\right) = 0$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{x}{y} = y}$$

$$P = \frac{1}{y}, \quad Q = y$$

$$I.F. = e^{\int P \cdot dy}$$

$$= e^{\int \frac{1}{y} \cdot dy}$$

$$= e^{\log_e y} = y$$

$$\boxed{a^{\log_a x} = x}$$

Solution: $\boxed{x(I.F) = \int Q \cdot (I.F) \cdot dy + c}$

$$\Rightarrow x(y) = \int (y)(y) \cdot dy + c$$

$$\Rightarrow xy = \int y^2 \cdot dy + c \quad \Rightarrow xy = \frac{y^3}{3} + c$$

$$\Rightarrow \boxed{x = \frac{y^2}{3} + \frac{c}{y}}$$

Q-12 $(x + 3y^2)\frac{dy}{dx} = y \quad (y > 0)$

$$\boxed{\frac{dx}{dy} + P.x = Q}$$

$$\Rightarrow (x + 3y^2) = y \cdot \frac{dx}{dy}$$

$$\boxed{P = -\frac{1}{y}, \quad Q = 3y}$$

$$\Rightarrow \left(y \cdot \frac{dx}{dy} - x = 3y^2\right) / y$$

$$I.F. = e^{\int P \cdot dy}$$

$$\Rightarrow \boxed{\frac{dx}{dy} - \frac{x}{y} = 3y}$$

$$= e^{-\int \frac{1}{y} \cdot dy}$$

$$= e^{(-\log y)}$$

$$n \log m = \log m^n$$

$$= \cancel{e}^{\log(y^{-1})} = y^{-1} = \frac{1}{y}$$

$$\boxed{x\,(I.F) = \int Q.\,(I.F).\,dy + c}$$

Solution:

$$x\left(\frac{1}{y}\right) = \int (3y).\left(\frac{1}{y}\right).\,dy + c$$

$$\Rightarrow \frac{x}{y} = \int 3.\,dy + c$$

$$\Rightarrow \frac{x}{y} = 3y + c$$

$$\Rightarrow \boxed{x = 3y^2 + cy}$$

Exercise 9.6 (Linear Differential Equations)

Find a particular solution satisfying the given condition

Q.13 $\frac{dy}{dx} + 2y\tan x = \sin x$; $y=0$ when $x = \frac{\pi}{3}$

L.D.E

$\boxed{\frac{dy}{dx} + P\cdot y = Q}$ (x)

$P = 2\tan x$

$Q = \sin x$

$I.F. = e^{\int P.dx} = e^{2\int \tan x . dx} = e^{2\log \sec x}$

$= e^{\log_e(\sec^2 x)}$

$= \sec^2 x = I.F$

$\boxed{a^{\log_a x} = x}$

$\boxed{\log m^n = n.\log m}$

Solution: $\boxed{y.(I.F) = \int Q.(I.F).dx + C}$

$\Rightarrow y.(\sec^2 x) = \int \sin x.(\sec^2 x).dx + C$

$\Rightarrow y.\sec^2 x = \int \sin x . \frac{1}{\cos^2 x}.dx + C$

$\Rightarrow y.\sec^2 x = \int (\tan x . \sec x) dx + C$

$\Rightarrow \boxed{y.\sec^2 x = \sec x + C}$

$y=0, x = \frac{\pi}{3}$

$0 = \sec\frac{\pi}{3} + C$

$\Rightarrow 0 = 2 + C$

$\Rightarrow \boxed{C = -2}$

$\frac{y}{\cos^2 x} = \frac{1}{\cos x} - 2$

$\Rightarrow \boxed{y = \cos x - 2\cos^2 x}$

$\boxed{Q.14}$ $(1+x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1+x^2}$; $y=0$ when $x=1$.

L.D.E.

$$\Rightarrow \boxed{\frac{dy}{dx} + \underbrace{\frac{2x}{1+x^2}}_{P} \cdot y = \underbrace{\frac{1}{(1+x^2)^2}}_{Q}} \equiv \boxed{\frac{dy}{dx} + Py = Q}$$

$$I.F. = e^{\int P \cdot dx} = e^{\int \frac{2x}{1+x^2} \cdot dx} = e^{\log(1+x^2)} = (1+x^2)$$

Solution: $\boxed{y \cdot (I.F) = \int Q \cdot (I.F) \cdot dx + C}$

$$\Rightarrow y(1+x^2) = \int \frac{1}{(1+x^2)^{\cancel{2}}} \cdot \cancel{(1+x^2)} \cdot dx + C$$

$$\Rightarrow y(1+x^2) = \int \frac{1}{(1+x^2)} \cdot dx + C$$

$$\Rightarrow \boxed{y(1+x^2) = \tan^{-1}(x) + C}$$

$x=1$, $y=0$

$0 \cdot (\ ) = \tan^{-1}(1) + c$

$\Rightarrow 0 = \dfrac{\pi}{4} + c$

$\Rightarrow \boxed{C = -\dfrac{\pi}{4}}$

Particular Solⁿ.

$$\Rightarrow \boxed{y(1+x^2) = \tan^{-1}(x) - \frac{\pi}{4}}$$

Q.15 $\frac{dy}{dx} - 3y\cot x = \sin 2x$; $y = 2$ when $x = \frac{\pi}{2}$

L.D.E

$\boxed{\frac{dy}{dx} + P.y = Q}$

$P = -3\cot x$

$Q = \sin 2x$

$I.F. = e^{\int P.dx} = e^{-3\int \cot x . dx} = e^{-3.\log(\sin x)}$

$I.F. = e^{\log(\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 x}$

Sol^n, $\boxed{y.(I.F) = \int Q.(I.F.).dx + C}$

$\Rightarrow y.\left(\frac{1}{\sin^3 x}\right) = \int \sin 2x . \frac{1}{\sin^3 x} . dx + C$

$\Rightarrow \frac{y}{\sin^3 x} = \int \frac{2\cancel{\sin x}.\cos x}{\cancel{\sin^3 x}\sin^2 x} . dx + C$

$\Rightarrow \frac{y}{\sin^3 x} = 2\int (\cot x . \operatorname{cosec} x).dx + C$

$\Rightarrow \boxed{\frac{y}{\sin^3 x} = 2(-\operatorname{cosec} x) + C}$

$y = 2, x = \frac{\pi}{2}$

$\frac{2}{\sin^3 \frac{\pi}{2}} = -2\operatorname{cosec}\frac{\pi}{2} + C$

$\Rightarrow 2 = -2 + C$

$\Rightarrow \boxed{4 = C}$

Particular Sol^n.

$\Rightarrow \frac{y}{\sin^3 x} = \frac{-2}{\sin x} + 4$

$\Rightarrow \boxed{y = -2\sin^2 x + 4\sin^3 x}$

Q.16 Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

(+)

Ans. ATQ Slope of the tangent at point (x, y) = Sum of the coordinates of the point (x, y)

$$\Rightarrow \boxed{\frac{dy}{dx} = x + y} \leftarrow \text{D.E.}$$

L.D.E.

$$\Rightarrow \boxed{\frac{dy}{dx} - y = x} \qquad \boxed{\frac{dy}{dx} + P.y = Q}$$

(x) - - -

$$\left.\begin{aligned} P &= -1 \\ Q &= x \end{aligned}\right\}$$

$$I.F. = e^{\int P.dx} = e^{-1\int dx} = e^{-x}$$

Solution. $\boxed{y.(I.F.) = \int Q(I.F.).dx + C}$

$$\Rightarrow y.e^{-x} = \int \overset{I}{x}.\overset{II}{e^{-x}}.dx + C$$

ILATE

(Int. by Parts)

$$\boxed{\int I.II = I\int II - \int\left(I'.\int II\right)}$$

$$\Rightarrow y.e^{-x} = \underbrace{x(-e^{-x}) - \int 1.(-e^{-x}).dx}+ C$$

$$\Rightarrow y.e^{-x} = -xe^{-x} + (-e^{-x}) + C$$

$$\Rightarrow y \cdot e^{-x} = \underline{-x \cdot e^{-x} - e^{-x}} + C$$

$$\Rightarrow y \cdot e^{-x} + xe^{-x} + e^{-x} = c$$

$$\Rightarrow \boxed{e^{-x}(y + x + 1) = c} \rightarrow \text{Passes through } (\overset{x}{0}, \overset{y}{0})$$

$e^{-0}(0+0+1) = c$

$\Rightarrow 1(1) = c$

$\boxed{c = 1}$

(1)

$$\Rightarrow e^{-x}(y+x+1) = 1$$

$$\Rightarrow \boxed{y + x + 1 = e^{x}}$$

Q.17 Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Point on the curve $= (x, y)$

ATQ Sum of Coordinates of (x, y) = Slope of tangent at (x, y) + 5

$$\Rightarrow x + y = \frac{dy}{dx} + 5$$

$$\Rightarrow x - 5 = \frac{dy}{dx} - y$$

$$\Rightarrow \boxed{\frac{dy}{dx} - y = x - 5}$$

L.D.E.

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$P = -1$

$Q = (x - 5)$

$\underline{P = -1}$, $\underline{Q = x-5}$

$\underline{I.F.} = e^{\int P.dx} = e^{-1\int dx} = \underline{e^{-x}}$

$\underline{\text{Sol}^n}$. $\boxed{y.(I.F) = \int Q.(I.F).dx + C}$

$\Rightarrow y.(e^{-x}) = \int \overset{I}{(x-5)}.\overset{II}{e^{-x}}.dx + C$

ILATE (Int. by Parts)

$\Rightarrow y.(e^{-x}) = (x-5).\int e^{-x}.dx - \int\left[(1-0).\underline{\int e^{-x}.dx}\right].dx + C$

$\Rightarrow y.(e^{-x}) = \underline{(x-5).(-e^{-x})} + \int e^{-x}.dx + C$

$\Rightarrow y.e^{-x} = -\underline{(x-5)e^{-x}} - \underline{e^{-x}} + C$

$\Rightarrow \boxed{y.e^{-x} = [(5-x).-1].e^{-x} + C}$

Passes through $(0,2)$

$x=0$
$y=2$

$2.e^{-0} = [4-0].e^{-0} + C$

$\Rightarrow 2 = 4 + C$

$\Rightarrow \boxed{-2 = C}$

$\Rightarrow \underline{y.e^{-x} = [4-x].e^{-x} - 2}$

$\Rightarrow \boxed{y = (4-x) - 2e^{x}}$

Q.18 The integrating factor of the differential eqn. $\underbrace{x\frac{dy}{dx} - y = 2x^2}_{(x)}$ is — (A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \underset{Q}{2x} \quad \xleftrightarrow{\text{L.D.E.}} \quad \boxed{\frac{dy}{dx} + Py = Q}$$

$P = -\frac{1}{x}$

$$\text{I.F.} = e^{\int P\cdot dx} = e^{-\int \frac{1}{x}\cdot dx}$$

$$= e^{-\log x} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$$

Q.19 The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay$, $(-1 < y < 1)$ is →

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Ans. $(1-y^2)\cdot\frac{dx}{dy} + yx = ay$

$$\Rightarrow \boxed{\frac{dx}{dy} + \left(\frac{y}{1-y^2}\right)x = \frac{ay}{1-y^2}} \rightarrow Q \quad = \quad \boxed{\frac{dx}{dy} + P\cdot x = Q} \text{ (L.D.E.)}$$

$(1-y^2)' = -2y$

$P = \frac{y}{1-y^2}$

$$\text{I.F.} = e^{\int P\cdot dy} = e^{\int \frac{y}{1-y^2}\cdot dy} = e^{\frac{1}{-2}\int \frac{-2y\,dy}{(1-y^2)}}$$

$-2y\,dy \rightarrow dt$, $1-y^2 \rightarrow t$

$$= e^{\left(-\frac{1}{2}\right)\log|1-y^2|} = \cancel{e}^{\cancel{\log}\,(1-y^2)^{-1/2}}$$

$$= (1-y^2)^{-1/2} = \frac{1}{(1-y^2)^{1/2}} = \frac{1}{\sqrt{1-y^2}}$$

Miscellaneous Exercise on Chapter 9

Q.1 Indicate its order & degree (if defined).

(i) $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$ ✓ Polynomial in Derivatives

Highest order Derivative

order of D.E. $= 2$ | Degree $= 1$ ✓

(ii) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$ ✓ Polynomial in Deri.

Highest order Derivative

| order $= 1$ | Degree $= 3$

(iii) $\left(\frac{d^4y}{dx^4}\right) - \sin\left(\frac{d^3y}{dx^3}\right) = 0$ ~~Not a~~ Polynomial in Derivatives

Highest order Derivative

| order $= 4$ ✓ | Degree $=$ Not Defined

Q.2 Verify that the given function (implicit or explicit) is a solution of the corresponding differential eqn.

(i) $xy = \underbrace{ae^{x} + be^{-x}} + x^2$: $\boxed{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0}$

a, b → Arbitrary Constants

$\Rightarrow \boxed{xy - x^2 = ae^{x} + be^{-x}}$ —— ①

diff. w.r.t. x →

$\Rightarrow y + x\frac{dy}{dx} - 2x = ae^{x} - be^{-x}$ —— ②

Again by Diff. w.r.t. x →

$\Rightarrow \boxed{\frac{dy}{dx} + \frac{dy}{dx} + x.\frac{d^2y}{dx^2} - 2 = ae^{x} + be^{-x}}$ —— ③

by eqn ① & ③ :⇒

$\Rightarrow \underbrace{x.\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2} = \underbrace{xy - x^2}$

$\Rightarrow \boxed{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0}$

H.P.

(ii) $y = e^x (a\cos x + b\sin x)$ ——— ① (u.v)' : $\boxed{\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0}$

$$\frac{dy}{dx} = \underbrace{e^x (a\cos x + b\sin x)}_{y \text{ (by eq}^n \text{ ①)}} + e^x(-a\sin x + b\cos x)$$

$$\frac{dy}{dx} = y + e^x(-a\sin x + b\cos x)$$

$$\Rightarrow \frac{dy}{dx} - y = e^x(-a\sin x + b\cos x) \quad \text{——— ②}$$

By diff. w.r.t. x →

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \underbrace{e^x(-a\sin x + b\cos x)}_{\frac{dy}{dx} - y \text{ by eq}^n \text{ ②}} + e^x(-a\cos x - b\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \left(\frac{dy}{dx} - y\right) - \underbrace{e^x(a\cos x + b\sin x)}_{y \text{ by eq}^n \text{ ①}}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - (y)$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0}$$

(iii) $y = x\sin 3x$: $\boxed{\dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0}$

Particular sol^n.

Free from Arbitrary Constants

$y = x\sin 3x$ $(u.v)'$

$$\frac{dy}{dx} = 1.\sin 3x + 3x.\cos 3x$$

$$\frac{d^2y}{dx^2} = 3\cos 3x + 3\cos 3x - 3x\sin 3x.(3)$$

$$\to = 6\cos 3x \cancel{+3\cos 3x} - 9x\sin 3x$$

$$LHS = \frac{d^2y}{dx^2} + 9y - 6\cos 3x$$

$$= (\cancel{6\cos 3x} - \cancel{9x\sin 3x}) + \cancel{9(x\sin 3x)} - \cancel{6\cos 3x}$$

$$= 0 = RHS.$$

(iv) $x^2 = 2y^2.\log y$: $\boxed{(x^2+y^2)\dfrac{dy}{dx} - xy = 0}$

diff. w.r.t x →

$$\Rightarrow \cancel{2}x = \cancel{4}y.y'.\log y + \cancel{2}y^2.\left(\frac{1}{y}\right).y'$$

$$\Rightarrow x = 2yy'.\log y + y.y'$$

$$\Rightarrow x = yy'.\{2\log y + 1\}$$

$$\because \boxed{\log y = \frac{x^2}{2y^2}}$$

$$\Rightarrow x = yy'\left(\cancel{2}.\frac{x^2}{\cancel{2}y^2} + 1\right)$$

$$\Rightarrow x = \cancel{y}y'\left(\frac{x^2+y^2}{\cancel{y^2}y}\right)$$

$$\Rightarrow xy = y'(x^2+y^2)$$

$$\Rightarrow \boxed{0 = \frac{dy}{dx}(x^2+y^2) - xy}$$

Miscellaneous Exercise on Chapter 9

Q.3 Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Diff. Eqⁿ — No. Arbi. Constants

Ans. $(x-a)^2 + 2y^2 = a^2$

$\Rightarrow x^2 - 2ax + \cancel{a^2} + 2y^2 = \cancel{a^2}$

$\Rightarrow x^2 + 2y^2 = 2ax$

$\Rightarrow \dfrac{(x^2+2y^2)}{x} = 2a$

Diff. w.r.t. x $\quad \left(\dfrac{u}{v}\right)' = \dfrac{u'.v - u.v'}{v^2}$

$\Rightarrow \dfrac{(2x + 4y.y').x - (x^2+2y^2).1}{x^2} = 0$

$\Rightarrow 2x^2 + 4xyy' - (x^2+2y^2) = 0$

$\Rightarrow 4xy\,y' = x^2 + 2y^2 - 2x^2$

$\Rightarrow \boxed{y' = \dfrac{2y^2 - x^2}{4xy}}$ ✓

Q.4 Prove that $(x^2-y^2) = c(x^2+y^2)^2$ is the general solution of differential equation $(x^3-3xy^2)dx = (y^3-3x^2y)dy$, where c is a parameter.

Ans. $\boxed{\dfrac{x^3-3xy^2}{y^3-3x^2y} = \dfrac{dy}{dx}} \longrightarrow$ H.D.E (Homo. Diff. eqⁿ)

$\downarrow$

$\dfrac{y}{x} = v$

$$\frac{dy}{dx} = \frac{(x^3 - 3xy^2)/x^3}{(y^3 - 3x^2y)/x^3} = \frac{1 - 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)}$$

Substitution, $\frac{y}{x} = v \Rightarrow y = x.v$

$$\Rightarrow \boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - \frac{v}{1} = \frac{1 \cancel{-3v^2} - v^4 \cancel{+3v^2}}{v^3-3v}$$

Variable separation

$$\Rightarrow \left(\frac{v^3-3v}{1-v^4}\right)dv = \frac{dx}{x}$$ integration

$$\Rightarrow \int\left(\frac{v^3}{1-v^4} - \frac{3v}{1-v^4}\right).dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{-4}\int\frac{\overbrace{-4.v^3\,dv}^{\to dt}}{\underbrace{1-v^4}_{\to t}} - \frac{3}{2}\int\frac{\overbrace{2v\,.dv}^{\to dm}}{1-(\underbrace{v^2}_{\to m})^2} = \int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{4}\int\frac{dt}{t} - \frac{3}{2}\int\frac{dm}{1-m^2} = \int\frac{dx}{x}$$

$$\Rightarrow \underbrace{\frac{-1}{4}\log(t)} - \underbrace{\frac{3}{2}\left(\frac{1}{2}\log\left|\frac{1+m}{1-m}\right|\right)} = \log x + \log c_1$$

$$\Rightarrow -\frac{1}{4}\left\{\log(1-v^4) + 3.\log\left|\frac{1+v^2}{1-v^2}\right|\right\} = \log|xc_1|$$

$$\Rightarrow \log(1-v^4) + \log\left(\frac{1+v^2}{1-v^2}\right)^3 = -4\log|x c_1|$$

$$\Rightarrow \log\left\{(1-v^4)\cdot\frac{(1+v^2)^3}{(1-v^2)^3}\right\} = \log(x c_1)^{-4}$$

$$\boxed{1-v^4 = (1-v^2)(1+v^2)} \quad (v^2)^2$$

$$\Rightarrow \frac{(1+v^2)\cancel{(1-v^2)}\cdot(1+v^2)^3}{\cancel{(1-v^2)^3}\ (1-v^2)^2} = \frac{1}{(x c_1)^4}$$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = \frac{1}{(x c_1)^4}$$

Square root

$$\boxed{v = \frac{y}{x}}$$

$$\Rightarrow \frac{(1+v^2)^2}{(1-v^2)} = \frac{1}{(x c_1)^2}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^2}{1-\frac{y^2}{x^2}} = \frac{1}{x^2\cdot c_1^2} \Rightarrow \frac{\left(\frac{x^2+y^2}{x^2}\right)^2}{\left(\frac{x^2-y^2}{x^2}\right)} = \frac{1}{x^2\cdot c_1^2}$$

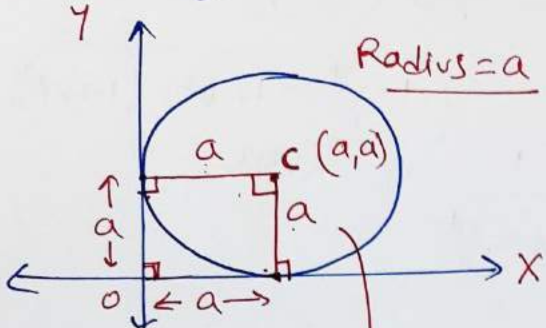
$$\Rightarrow \frac{(x^2+y^2)^2}{\cancel{x^4}\ \cancel{x^2}} \times \frac{\cancel{x^2}}{(x^2-y^2)} = \frac{1}{\cancel{x^2}\cdot c_1^2}$$

$$\Rightarrow c_1^2\cdot(x^2+y^2)^2 = (x^2-y^2)$$

$$\boxed{C\cdot(x^2+y^2)^2 = (x^2-y^2)}$$

Q.5 Form the differential equation of the family of circles in the first quadrant which touch the coordinates axes.

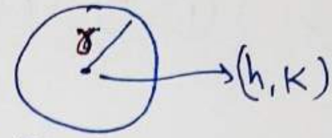
$\Downarrow$



Radius $= a$
Centre $= (a, a)$

$\longrightarrow$ $\underline{Eq^n}$ Family of circles

$$\boxed{(x-a)^2+(y-a)^2 = a^2}$$



$\underline{Eq^n}$ $(x-h)^2+(y-k)^2 = r^2$

a = Arbitrary Constant.

Diff. eq^n $\rightarrow$ Free from Arb. Const.

order = 1 $\leftarrow$ 1 बार की Diff.

$$(x-a)^2+(y-a)^2 = a^2 \quad \text{—} \ (1)$$

by Diff. w.r.t. x

$$\Rightarrow \cancel{2}(x-a)+\cancel{2}(y-a)\cdot y' = 0$$

$$\Rightarrow x - a + yy' - ay' = 0$$

$$\Rightarrow x + yy' = a + ay'$$

$$\Rightarrow x + yy' = a(1+y')$$

$$\Rightarrow a = \left(\frac{x+yy'}{1+y'}\right) \quad \text{—} \ (2)$$

by putting the value of 'a' from eq^n (2) to eq^n (1) $\rightarrow$

$$\Rightarrow \left(x - \frac{x+yy'}{1+y'}\right)^2 + \left(y - \frac{x+yy'}{1+y'}\right)^2 = \left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{\cancel{x}+xy'-\cancel{x}-yy'}{1+y'}\right)^2 + \left(\frac{y+\cancel{yy'}-x-\cancel{yy'}}{1+y'}\right)^2 = \left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{xy' - yy'}{\cancel{1+y'}}\right)^2 + \left(\frac{y-x}{\cancel{1+y'}}\right)^2 = \left(\frac{x+yy'}{\cancel{1+y'}}\right)^2$$

$$\Rightarrow (y')^2 (x-y)^2 + (y-x)^2 = (x+yy')^2$$

$$\boxed{\boxed{(a-b)^2 = (b-a)^2} \quad (4)^2 = 16 \;,\; (-4)^2 = 16}$$

$$\Rightarrow (y')^2 \cdot (x-y)^2 + (x-y)^2 = (x+yy')^2$$

$$\Rightarrow \boxed{(x-y)^2 \left((y')^2 + 1\right) = (x+yy')^2}$$

Miscellaneous Exercise on Chapter 9

Q.6 Find the general solution of the differential equation

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0.$$

[(1) Variable Separation
(2) Homo. D.E.
(3) Linear D.E.]

Solⁿ. $\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Variable separation →

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$ (integration)

$$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}} \Rightarrow \sin^{-1}y = -\sin^{-1}x + C$$

$$\Rightarrow \boxed{\sin^{-1}y + \sin^{-1}x = C}$$

Q.7 Show that general solution of the differential equation $\boxed{\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0}$ is given by $(x+y+1) = A(1-x-y-2xy)$, where A is parameter.

General Solution,

Ans. $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

variable separation

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

integration

$$\Rightarrow \boxed{\int \frac{dy}{y^2+y+1} = -\int \frac{dx}{x^2+x+1}}$$ (1)

$$I = \int \frac{dx}{x^2+x+1}$$

$$I = \int \frac{dx}{x^2+x+1}$$ (Completing the square method)

$$I = \int \frac{dx}{\underbrace{x^2+x+\left(\frac{1}{2}\right)^2}-\left(\frac{1}{2}\right)^2+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 \underbrace{-\frac{1}{4}+1}}$$

$a^2+2ab+b^2=(a+b)^2$

$$\frac{-1+4}{4} = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$I = \int \frac{dx}{\underset{\substack{\uparrow\\ X}}{\left(x+\frac{1}{2}\right)^2}+\underset{\substack{\uparrow\\ A}}{\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$\boxed{\int \frac{dX}{A^2+X^2} = \frac{1}{A}\tan^{-1}\left(\frac{X}{A}\right)+c}$$

$$I = \frac{1}{\left(\sqrt{3}/2\right)} \cdot \tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Similar $\int \frac{dx}{y^2+y+1} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)$

eqn. (1) $\rightarrow$ $\int \frac{dy}{y^2+y+1} = -\int \frac{dx}{x^2+x+1}$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\left\{\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right\} = C$$

$$\Rightarrow \tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\Rightarrow \tan\left(\underbrace{\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)}_{A} + \underbrace{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}_{B}\right) = \tan\left(\frac{\sqrt{3}}{2}c\right)$$

by taking 'tan' to both sides.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\Rightarrow \frac{\left(\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}\right)}{\left(\frac{1}{1} - \left(\frac{2y+1}{\sqrt{3}}\right)\cdot\left(\frac{2x+1}{\sqrt{3}}\right)\right)} = \tan\left(\frac{\sqrt{3}}{2}c\right)$$

$$\Rightarrow \frac{\left(\frac{2x+2y+2}{\cancel{\sqrt{3}}}\right)}{\left(\frac{3-(4xy+2y+2x+1)}{\cancel{3}\ \sqrt{3}}\right)} = \tan\left(\frac{\sqrt{3}}{2}c\right)$$

$$\Rightarrow \sqrt{3}\left(\frac{2(x+y+1)}{2-4xy-2y-2x}\right) = \tan\left(\frac{\sqrt{3}}{2}c\right)$$

$$\Rightarrow \frac{\cancel{2}(x+y+1)}{\cancel{2}(1-2xy-y-x)} = \frac{\tan\left(\frac{\sqrt{3}}{2}c\right)}{\sqrt{3}} = A$$

→ New Name

Parameter

$$\Rightarrow \boxed{(x+y+1) = A(1-2xy-y-x)}$$

Q.8 Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is

$\sin x \cos y . dx + \cos x \sin y . dy = 0.$

Ans. $\sin x . \cos y . dx + \cos x . \sin y \, dy = 0$

$\Rightarrow \sin x . \cos y . dx = -\cos x . \sin y . dy$

V.S.

$\Rightarrow \frac{\sin x}{\cos x} . dx = -\frac{\sin y \, dy}{\cos y}$

integration.

$\Rightarrow \int \tan x . dx = -\int \tan y . dy$

$\Rightarrow \log(\sec x) = -\log(\sec y) + \log . c$

$\Rightarrow \log(\sec x) + \log(\sec y) = \log c$

$\log m + \log n = \log mn$

$\Rightarrow \cancel{\log}(\sec x . \sec y) = \cancel{\log} . c$

$\Rightarrow \boxed{\sec x . \sec y = c}$

Curve passes through $\left(\overset{x}{0}, \overset{y}{\frac{\pi}{4}}\right)$

$\sec 0 . \sec \frac{\pi}{4} = c$

$1 . \sqrt{2} = c$

$c = \sqrt{2}$

$\Rightarrow \boxed{\sec x . \sec y = \sqrt{2}}$

$\Rightarrow \frac{\sec x}{\sqrt{2}} = \frac{1}{\sec y}$

$\Rightarrow \boxed{\frac{\sec x}{\sqrt{2}} = \cos y}$

Q.9 Find the particular solution of the differential equation $(1+e^{2x})\,dy + (1+y^2)e^x\,dx = 0$; given that $y=1$, $x=0$.

Ans: $(1+e^{2x}).dy = -(1+y^2).e^x dx$

variable separation.

$\Rightarrow \dfrac{dy}{1+y^2} = -\dfrac{e^x dx}{1+e^{2x}}$

by integration.

$\Rightarrow \displaystyle\int \frac{dy}{1+y^2} = -\int \frac{e^x . dx}{1+(e^x)^2}$

formula

let $e^x = t$

$e^x . dx = dt$

$\Rightarrow \tan^{-1} y = -\displaystyle\int \frac{dt}{1+t^2}$

$\Rightarrow \tan^{-1} y = -\tan^{-1}(t) + c$

$\Rightarrow \boxed{\tan^{-1}(y) + \tan^{-1}(t) = c}$

e^x

$\Rightarrow \boxed{\tan^{-1}(y) + \tan^{-1}(e^x) = c}$

$y=1$, $x=0$ put. ?

$\Rightarrow \tan^{-1}(1) + \tan^{-1}(e^0) = c$

$\Rightarrow \dfrac{\pi}{4} + \tan^{-1}(1) = c$

$\dfrac{\pi}{4}$

$\Rightarrow \boxed{\dfrac{\pi}{2} = c}$

Particular solⁿ,

$\tan^{-1}(y) + \tan^{-1}(e^x) = \dfrac{\pi}{2}$

Miscellaneous Exercise on Chapter 9 →

Q.10 Solve the differential equation $y \cdot e^{\frac{x}{y}} dx = \left(x e^{\frac{x}{y}} + y^2\right) dy$.

Ans. $$\frac{dx}{dy} = \frac{x e^{x/y} + y^2}{y \cdot e^{x/y}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{x/y}}$$

Substitution

$$\boxed{\frac{x}{y} = V}$$

$$x = y.V$$

$$\boxed{\frac{dx}{dy} = V + y.\frac{dv}{dy}}$$

After Substitution :

$$\Rightarrow \cancel{V} + y\frac{dv}{dy} = \cancel{V} + \frac{y}{e^v}$$

Variable separation

$$\Rightarrow \cancel{y}\frac{dv}{dy} = \frac{\cancel{y}}{e^v} \Rightarrow \frac{dv}{dy} = \frac{1}{e^v} \Rightarrow e^v.dv = dy$$

integration: $$\Rightarrow \int e^v.dv = \int dy$$

$$\Rightarrow e^v = y + c$$

$$\Rightarrow \boxed{e^{x/y} = y + c}$$

Q.11 Find a particular solution of the differential equation $(x-y)(dx+dy) = (dx-dy)$ given that $y=-1$, when $x=0$.

$\boxed{\text{Hint : put } (x-y=t)}$.

Ans. $x-y=t$

diff. w.r.t. (x)

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \left(1 - \frac{dt}{dx}\right) = \frac{dy}{dx}$$

$$\frac{(x-y)(dx+dy)}{dx} = \frac{(dx-dy)}{dx}$$

$$\Rightarrow (x-y).\left(1+\frac{dy}{dx}\right) = \left(1-\frac{dy}{dx}\right)$$

$$\Rightarrow (x-y)\left(1+\frac{dy}{dx}\right) = \left(1-\frac{dy}{dx}\right)$$

$\boxed{\text{Substitute}}$ $\boxed{x-y=t}$ ✓

$$\boxed{\frac{dy}{dx} = 1-\frac{dt}{dx}}$$ ✓

$$\Rightarrow (t)\left(1+1-\frac{dt}{dx}\right) = 1-\left(1-\frac{dt}{dx}\right)$$

$$\Rightarrow 2t - t\cdot\frac{dt}{dx} = \cancel{1}-\cancel{1}+\frac{dt}{dx} \Rightarrow 2t = t\frac{dt}{dx}+\frac{dt}{dx}$$

$$\Rightarrow 2t = (t+1)\frac{dt}{dx} \Rightarrow dx = \left(\frac{t+1}{2t}\right)dt$$

integration: $\int dx = \int\left(\frac{1}{2}+\frac{1}{2t}\right)\cdot dt$

$$\Rightarrow x = \frac{t}{2}+\frac{1}{2}\cdot\log|t| + c$$

$$\Rightarrow \boxed{x = \frac{x-y}{2}+\frac{1}{2}\log|x-y| + c}$$

$y=-1,\ x=0$ put

$$\Rightarrow 0 = \frac{1}{2}+\underbrace{\frac{1}{2}\cancel{\log|1|}}_{}\,^{\to 0} + c \Rightarrow \boxed{c=-\frac{1}{2}}$$

Particular solⁿ: $x = \frac{x-y}{2}+\underbrace{\frac{1}{2}\log|x-y|}_{}-\frac{1}{2}$

$$\Rightarrow \frac{x}{1}-\left(\frac{x-y}{2}\right)+\frac{1}{2} = \frac{1}{2}\log|x-y|$$

$$\Rightarrow \frac{2x-x+y+1}{\cancel{2}} = \frac{1}{\cancel{2}}\log(x-y)$$

$$\Rightarrow \boxed{(x+y+1) = \log|x-y|}$$ ✓

Q.12 Solve the differential eqⁿ: $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy} = 1 \cdot (x \neq 0)$

L.D.E.

$$\boxed{\frac{dy}{dx} + P.y = Q}$$

Standard

x के fn.

$$\Rightarrow \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\Rightarrow \frac{e^{-2\sqrt{x}}}{\sqrt{x}} = \frac{dy}{dx} + \frac{y}{\sqrt{x}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}}$$

$$\left.\begin{aligned} P &= \frac{1}{\sqrt{x}} \\ Q &= \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \end{aligned}\right\}$$

Integrating factor

I.F.

$$= e^{\int P.dx} = e^{\int \frac{1}{\sqrt{x}}\cdot dx} = e^{\int x^{-1/2}\cdot dx}$$

$$= e^{\left(\frac{x^{-1/2+1}}{-1/2+1}\right)} = e^{\left(\frac{x^{1/2}}{1/2}\right)} = e^{2\sqrt{x}} = \text{I.F.}$$

Solution:

$$\boxed{y.(\text{I.F}) = \int Q.(\text{I.F}).dx + C}$$

$$\Rightarrow y.\left(e^{2\sqrt{x}}\right) = \int \frac{\cancel{e^{-2\sqrt{x}}}}{\sqrt{x}}\cdot \cancel{e^{2\sqrt{x}}}\,.dx + C$$

$$\Rightarrow y.\left(e^{2\sqrt{x}}\right) = \int \frac{1}{\sqrt{x}}\cdot dx + C$$

$$\Rightarrow \boxed{y.\left(e^{2\sqrt{x}}\right) = 2\sqrt{x} + C}$$

Q.13 Find a particular solution of the diff. eqn.

$$\frac{dy}{dx} + y.\cot x = 4x\,\text{cosec}\,x, \quad (x \neq 0), \text{ given that } y=0,\ x=\frac{\pi}{2}.$$

L.D.E. $\boxed{\frac{dy}{dx} + P.y = Q}$

$P = \cot x$

$Q = 4x\,\text{cosec}\,x$

$$\text{I.F.} = e^{\int P.dx}$$

$$= e^{\int \cot x . dx} = e^{\log|\sin x|} = \sin x$$

Solution: $\boxed{y.(\text{IF}) = \int Q.(\text{I.F}).dx + C}$

$$\Rightarrow y.\sin x = \int 4x\,\cancel{\text{cosec}\,x}.\cancel{\sin x}.dx + C$$

$$\Rightarrow y.\sin x = \cancel{4}^{2}.\frac{x^2}{\cancel{2}} + C$$

$$\Rightarrow \boxed{y\sin x = 2x^2 + C}$$ General soln.

$y=0,\ x=\frac{\pi}{2}$

$$\Rightarrow 0.\left(\sin\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right)^2 + C$$

$$\Rightarrow -2.\frac{\pi^2}{4} = C$$

$$\Rightarrow \boxed{-\frac{\pi^2}{2} = C}$$

Particular soln.

$$\boxed{y\sin x = 2x^2 - \frac{\pi^2}{2}}$$

Miscellaneous Exercise on Chapter 9

Q.14 Find a particular solution of the diff. eq^n.

$(x+1)\dfrac{dy}{dx} = 2e^{-y}-1$, given that $y=0$ when $x=0$.

Ans. $(x+1)\dfrac{dy}{dx} = \left(2e^{-y}-1\right)$

1. Variable separation
2. Homo. Diff. Eq^n.
3. Linear Diff. Eq^n.

$$\Rightarrow \frac{dy}{\left(2e^{-y}-1\right)} = \frac{dx}{(x+1)}$$

integration

$$\Rightarrow \int \frac{dy}{\left(\frac{2}{e^y}-1\right)} = \int \frac{dx}{(x+1)} \Rightarrow \int \frac{dy}{\left(\frac{2-e^y}{e^y}\right)} = \int \frac{dx}{x+1}$$

$$\Rightarrow -\int \frac{-e^y\,dy}{\underbrace{2-e^y}_{t}} = \int \frac{dx}{x+1} \qquad (-e^y\,dy \to dt)$$

$\log mn = \log m + \log n$

$$\Rightarrow -\int \frac{dt}{t} = \int \frac{dx}{x+1} \Rightarrow -\log|t| = \log|x+1| + \log c$$

$n.\log m = \log m^n$

$$\Rightarrow -\log\left(2-e^y\right) = \log\left[(x+1).c\right]$$

$$\Rightarrow \log\left(2-e^y\right)^{-1} = \log\left[c.(x+1)\right]$$

$$\Rightarrow \boxed{\frac{1}{2-e^y} = c.(x+1)}$$

$y=0,\ x=0$

$$\Rightarrow \frac{1}{2-e^0} = c.(1)$$

$$\Rightarrow \frac{1}{2-1} = c$$

$$\Rightarrow \boxed{c=1}$$

Particular Solution

$$\frac{1}{2-e^y} = (1)(x+1) \Rightarrow \frac{1}{x+1} = 2-e^y$$

$C=1$

$$\Rightarrow e^y = 2 - \frac{1}{x+1} \Rightarrow e^y = \frac{2x+2-1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+1}{x+1} \Rightarrow \boxed{y = \log\left(\frac{2x+1}{x+1}\right)}$$

Exponential fn^n $\Leftrightarrow$ log

Q.15 The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Ans.

	(t) Time Duration	Time line (years)	Population
Case – I	$t=0$	1999	$P=20000$
Case – II	$t=5$	2004	$P=25000$
Case – III	$t=10$	2009	$P=?$

Derivative

Rate of change of population $= \dfrac{dP}{dt}$

Let 'P' be populat^n of village at some time

ATQ. $\dfrac{dP}{dt} \propto P$

$$\Rightarrow \boxed{\frac{dP}{dt} = K.P} \rightarrow \text{Diff. eq}^n.$$

$K =$ ~~Pro~~ proportionality Constant

$$\Rightarrow \boxed{\frac{dP}{dt} = KP}$$

by variable separation:

$$\Rightarrow \frac{dP}{P} = K\,dt$$

integration

Const. Const.

$$\Rightarrow \int \frac{dP}{P} = K \int dt \Rightarrow \boxed{\log(P) = Kt + C}$$

Case-I $t = 0,\ P = 20000$

$$\Rightarrow \log 20000 = \cancel{K(0)}^{0} + C$$

$$\Rightarrow \boxed{C = \log 20000}$$

updated, $\boxed{\log P = Kt + \log 20000}$

Case-II. $t = 5,\ P = 25000$

$$\Rightarrow \log(25000) = K(5) + \log(20000)$$

$$\Rightarrow \log\left(\frac{\cancel{25000}^{5}}{\cancel{20000}_{4}}\right) = 5.K \Rightarrow \boxed{K = \frac{1}{5}\log\left(\frac{5}{4}\right)}$$

updated $\boxed{\log P = \frac{1}{5}\log\left(\frac{5}{4}\right).t + \log 20000}$

$t =$ 10 years (in 2009)

$$\Rightarrow \log(P) = \frac{1}{\cancel{5}}\boxed{\log\left(\frac{5}{4}\right)}.\cancel{10}^{2} + \log(20000)$$

$$\Rightarrow \log P = 2.\log\left(\frac{5}{4}\right) + \log 20000$$

$$\Rightarrow \log P = \log\left(\frac{5}{4}\right)^2 + \log 20000$$

$$\Rightarrow \cancel{\log} P = \cancel{\log}\left[\frac{5 \times 5 \times 20000}{4 \times 4}\right]$$

$$\Rightarrow \boxed{P = 31250}$$

Q.16 The general solution of the diff. eqⁿ.

$$\boxed{\frac{y\,dx - x\,dy}{y} = 0}$$ is — (A) $xy = c$ (B) $x = cy^2$ (C) $y = cx$ (D) $y = cx^2$

$\Rightarrow y\,dx - x\,dy = 0$

$\Rightarrow y.dx = x.dy$

variable separation,

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

Int. $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\Rightarrow \log x = \log y + \log c_1$

$\Rightarrow \cancel{\log} x = \cancel{\log}(y.c_1)$

$\Rightarrow x = y.c_1$

$\Rightarrow y = \frac{x}{c_1} \Rightarrow \boxed{y = cx}$

Q.17 → Directly in Book

Q.18 The general solution of the differential eqⁿ.

$e^x dy + (ye^x + 2x)dx = 0$ is — (A) $x.e^y + x^2 = c$ (B) $xe^y + y^2 = c$ (C) $ye^x + x^2 = c$ (D) $ye^y + x^2 = c$

L.D.E. $\boxed{\frac{dy}{dx} + P.y = Q}$

$$\frac{e^x dy + (ye^x + 2x)dx}{dx} = \frac{0}{dx}$$

$\Rightarrow e^x.\frac{dy}{dx} + ye^x + 2x = 0$

$\Rightarrow \boxed{\frac{dy}{dx} + 1.y = \frac{-2x}{e^x}}$

$P = 1,\ Q$

$$\boxed{\begin{aligned} I.F. &= e^{\int P.dx} \\ &= e^{\int 1.dx} = e^x \end{aligned}}$$

Solution,

$y.(I.F) = \int Q(I.F).dx + c$

$\Rightarrow y(e^x) = \int \frac{-2x}{\cancel{e^x}}.\cancel{e^x}.dx + c$

$\Rightarrow y.e^x = -x^2 + c$

$\Rightarrow \boxed{y.e^x + x^2 = c}$