

[Inverse Trigonometric Functions]

(प्रतिलोम त्रिकोणमितीय फलन)

Inverse function

$$f: A \rightarrow B$$

Bijjective

One-one	onto
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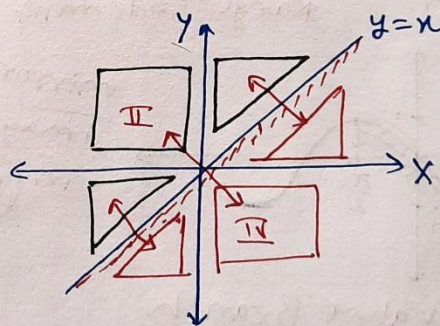
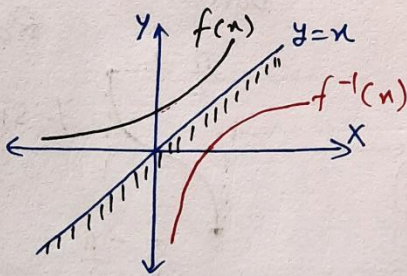
$$f^{-1}: B \rightarrow A$$

Bijjective

one-one	onto
---------	------

- Domain of $f = A =$ Range of f^{-1}
- Range of $f = B =$ Domain of f^{-1}

- $f^{-1} \circ f(x) = x, x \in A$
- $f \circ f^{-1}(x) = x, x \in B$
- Graphs of f and f^{-1} are mirror image of each other in the mirror line $y=x$.

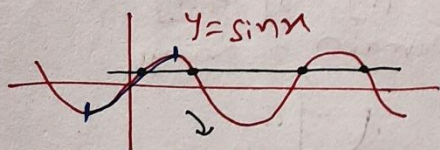


Trigonometric Functions \rightarrow

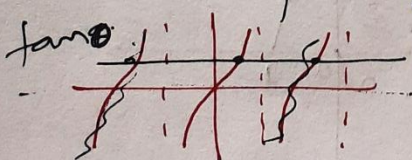
$$f(x) = y = \sin x$$

Real value
(output)

Angle (input)



many-one

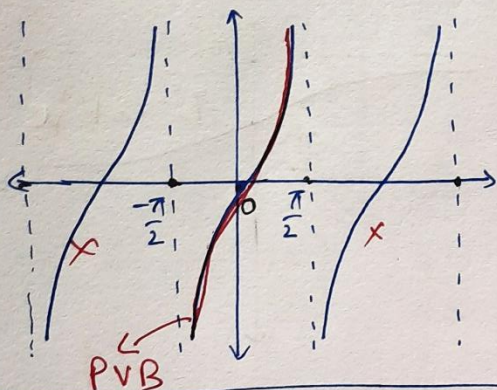


- Trigonometric Functions are not one-one in their Domain.
- But these functions can be made one-one - onto by modifying Domain & Domain.

3

$$y = f(x) = \tan x$$

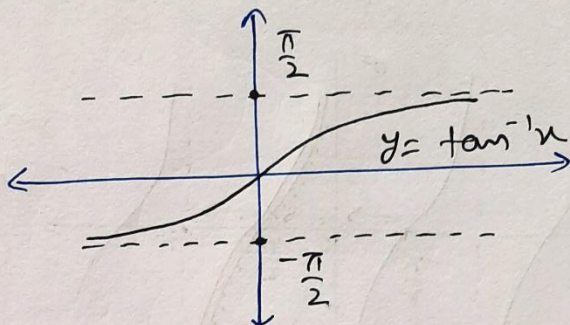
$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-\infty, \infty) \quad \text{R}$$



Mirror image

$$y = f^{-1}(x) = \tan^{-1} x$$

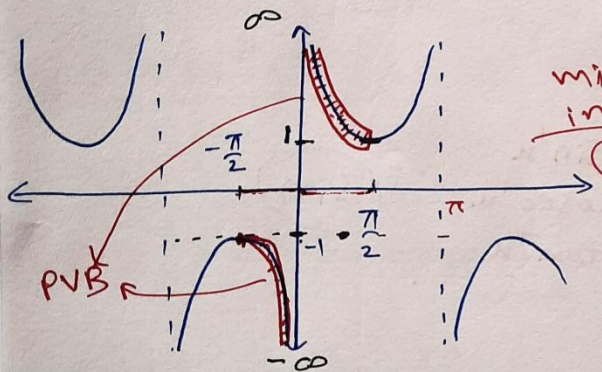
$$\mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



4

$$y = f(x) = \operatorname{cosec} x$$

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \rightarrow (-\infty, -1] \cup [1, \infty)$$

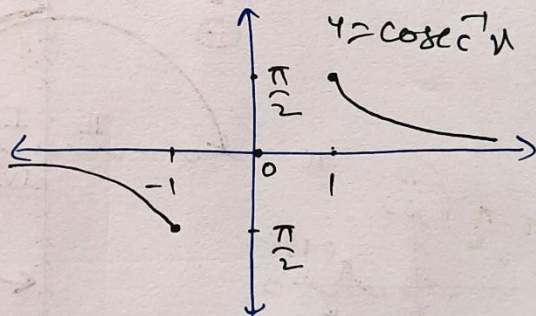


minor image
 $(y=x)$

$$y = f^{-1}(x) = \operatorname{cosec}^{-1} x$$

$$(-\infty, -1] \cup [1, \infty) \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

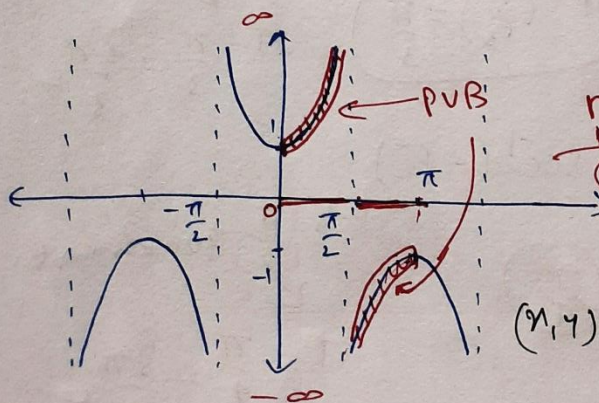
Domain Range



5

$$y = f(x) = \sec x$$

$$[0, \pi] - \left\{\frac{\pi}{2}\right\} \rightarrow (-\infty, -1] \cup [1, \infty)$$

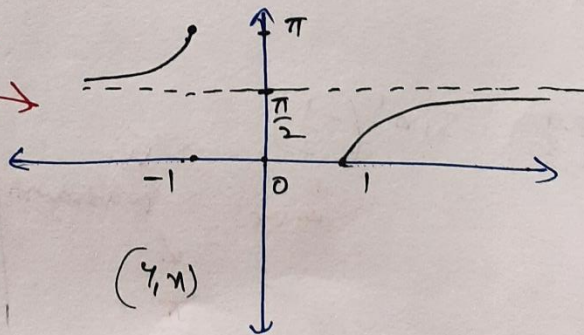


minor image
 $(y=x)$

$$y = f^{-1}(x) = \sec^{-1} x$$

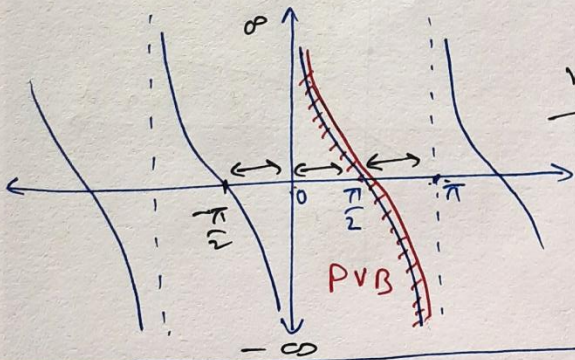
$$(-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

Domain Range



6 $y = \cot x = f(x)$

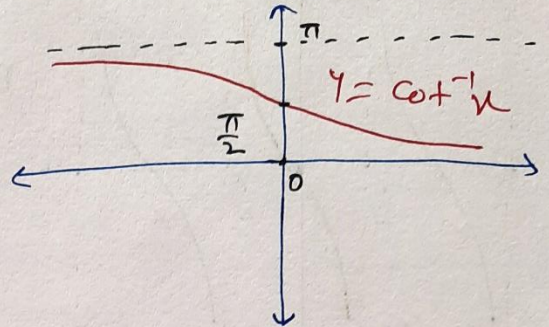
$(0, \pi) \rightarrow (-\infty, \infty)$
(R)



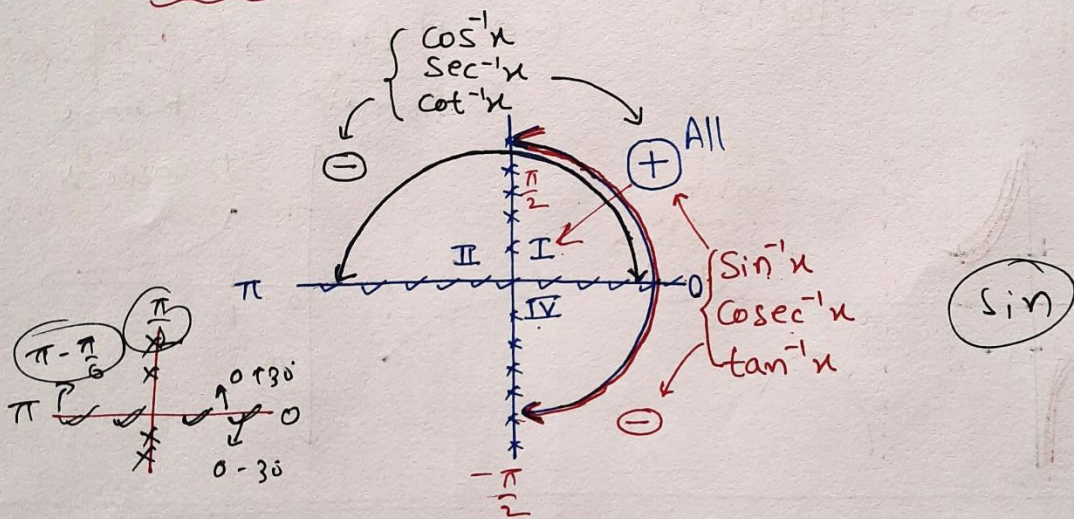
mirror image
 $y=x$

$y = \cot^{-1} x = f^{-1}(x)$

$(-\infty, \infty) \rightarrow (0, \pi)$
R Domain



How to Solve Questions Fast??



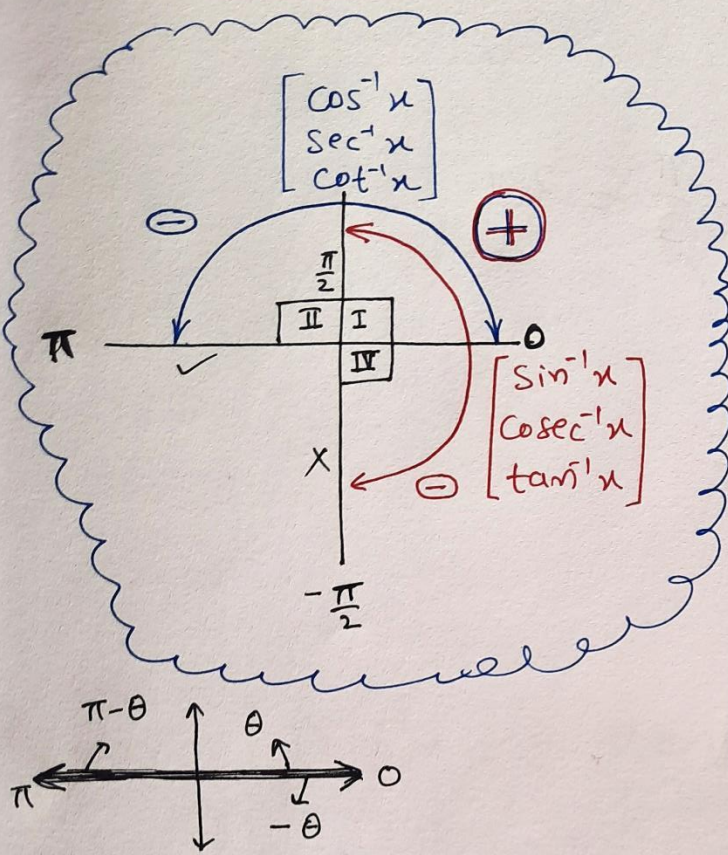
e.g. $\sin^{-1}(\frac{1}{2}) = x = \frac{\pi}{6}$
 $\frac{1}{2} = \sin(x)$
I

e.g. $\cot^{-1}(-\frac{1}{\sqrt{3}}) \rightarrow$ II-quadrant
 $\cot^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{3}$
 $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

e.g. $\sin^{-1}(-\frac{1}{2}) \rightarrow$ IV-quadrant
ve
 $-\frac{\pi}{6}$

e.g. $\sec^{-1}(-\sqrt{2}) \rightarrow$ II-quadrant
 $\frac{3\pi}{4}$
 $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Exercise 2.1



<u>ITF</u>	<u>Range</u> $\rightarrow y = \sin^{-1}x$
<u>$\sin^{-1}x$</u>	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
<u>\cos^{-1}</u>	$[0, \pi]$
<u>\tan^{-1}</u>	$(-\frac{\pi}{2}, \frac{\pi}{2})$
<u>$\operatorname{cosec}^{-1}$</u>	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$
<u>\sec^{-1}</u>	$[0, \pi] - \{\frac{\pi}{2}\}$
<u>\cot^{-1}</u>	$(0, \pi)$

Exercise 2.1 (ITF)

Q.1.

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} = \text{output (angle)}$$

↓
IV quadrant

Q.2

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

↓
I-quadrant

Q.3

$$\operatorname{cosec}^{-1}(2) = \frac{\pi}{6}$$

↓
I-quad.

$$\boxed{\text{Q. 4}} \quad \underbrace{\tan^{-1}(-\sqrt{3})}_{\text{IV-quad.}} = -\frac{\pi}{3}$$

$$\boxed{\text{Q. 5.}} \quad \underbrace{\cos^{-1}\left(-\frac{1}{2}\right)}_{\text{II-quad.}} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\boxed{\text{Q. 6}} \quad \underbrace{\tan^{-1}(-1)}_{\text{IV quad.}} = -\frac{\pi}{4}$$

$$\boxed{\text{Q. 7}} \quad \underbrace{\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)}_{\text{I-quad.}} = \frac{\pi}{6}$$

$$\boxed{\text{Q. 8}} \quad \underbrace{\cot^{-1}(\sqrt{3})}_{\text{I-quad.}} = \frac{\pi}{6}$$

$$\boxed{\text{Q. 9}} \quad \underbrace{\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{\text{II-quad.}} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\boxed{\text{Q. 10}} \quad \underbrace{\operatorname{cosec}^{-1}(-\sqrt{2})}_{\text{IV quad.}} = -\frac{\pi}{4}$$

$$\begin{aligned} \text{Q.11} \quad & \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) \\ &= \frac{\pi}{4} + \left(\pi - \frac{\pi}{3}\right) + \left(-\frac{\pi}{6}\right) \\ &= \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{Q.12} \quad & \cos^{-1}\left(\frac{1}{2}\right) + 2 \times \sin^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{2\pi}{3} \end{aligned}$$

Q.13 If $\sin^{-1} x = y$, then —

- (A) $0 \leq y \leq \pi$ ~~(B)~~ $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 (C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$y = \sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Q.14 $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ is equal to —

- (A) π ~~(B)~~ $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Ans. $\tan^{-1}(\sqrt{3}) - [\sec^{-1}(-2)]$

$$\begin{aligned} &= \frac{\pi}{3} - \left[\pi - \frac{\pi}{3}\right] = \frac{\pi}{3} - \pi + \frac{\pi}{3} \\ &= \frac{2\pi}{3} - \pi = -\frac{\pi}{3} \end{aligned}$$

Properties of Inverse Trigonometric Functions

Formula Sheet

Formula Hint

① (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$

~~tan~~
trigo⁻¹(1/x) →

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

② (i) $\sin^{-1}(-x) = -\sin^{-1}x$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

trigo⁻¹(-x) →

$$\sin(-\theta) = -\sin\theta$$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

$$\cos(\pi - \theta) = -\cos\theta$$

③ (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

(Complementary Angle)

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

④ (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$$

⑤ (i) $2 \tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(ii) $2 \tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

* (iii) $2 \tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

all are in their respective Domains

Proof:

$$\textcircled{1} \text{ (i)} \quad \sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x; \quad x \in (-\infty, -1] \cup [1, \infty)$$

$$\text{let } \operatorname{cosec}^{-1} x = \theta = \text{RHS}$$

$$\Rightarrow x = \operatorname{cosec} \theta$$

$$\Rightarrow \frac{1}{x} = \sin \theta$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \theta = \text{LHS}$$

$$\textcircled{2} \text{ (i)} \quad \sin^{-1}(-x) = -\sin^{-1} x$$

$$\text{LHS} = \sin^{-1}(-x) = \theta \text{ (let)}$$

$$\Rightarrow -x = \sin \theta$$

$$\Rightarrow x = -\sin \theta$$

$$\Rightarrow x = \sin(-\theta)$$

$$\Rightarrow \sin^{-1}(x) = -\theta$$

$$\Rightarrow \underline{-\sin^{-1} x = \theta}$$

$$\underline{\text{RHS} = \theta}$$

$$\text{(iv)} \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\text{LHS} = \theta = \cos^{-1}(-x)$$

$$\Rightarrow \cos \theta = -x$$

$$\Rightarrow -\cos \theta = x$$

$$\Rightarrow \cos(\pi - \theta) = x$$

$$\Rightarrow \pi - \theta = \cos^{-1} x$$

$$\Rightarrow \underline{\pi - \cos^{-1} x = \theta}$$

$$\underline{\text{RHS} = \theta}$$

$$\textcircled{3} \text{ (i)} \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\text{let } \sin^{-1} x = \theta$$

$$\Rightarrow x = \sin \theta$$

$$\Rightarrow x = \underline{\cos\left(\frac{\pi}{2} - \theta\right)}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$$

$$\Rightarrow \theta + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \underline{\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}}$$

$$\textcircled{4} \text{ (i) } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$$

let $\angle A$ $\angle B$

$$\tan^{-1} x = \textcircled{A}, \quad \tan^{-1} y = \textcircled{B}$$

$$\Rightarrow x = \textcircled{\tan A}, \quad y = \textcircled{\tan B}$$

we know that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\Rightarrow \tan(\tan^{-1} x + \tan^{-1} y) = \frac{x+y}{1-xy}$$

$$\Rightarrow \boxed{\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)} \star$$

$$\textcircled{5} \text{ (i) } \underline{2 \tan^{-1} x} = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

let $\tan^{-1} x = \theta \Rightarrow x = \tan \theta$

we know \rightarrow

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\Rightarrow \sin(\underline{2 \tan^{-1} x}) = \frac{2x}{1+x^2}$$

$$\Rightarrow \boxed{2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right)}$$

e.g. Show that $\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{2}{11}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$$\text{LHS} = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{2}{11}\right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) \quad \left\{ \begin{array}{l} \tan^{-1}(-x) \\ = -\tan^{-1}x \end{array} \right\}$$

$$\boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

$$= \tan^{-1}\left(\frac{\frac{1}{2} + \frac{2}{11}}{1 - \frac{1}{2} \times \frac{2}{11}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{11+4}{2 \times 11}}{\frac{22-2}{2 \times 11}}\right) = \tan^{-1}\left(\frac{\frac{15}{22}}{\frac{20}{22}}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) = \text{RHS}$$

Exercise 2.2(Chapter 2 - ITF)

Q.1 Prove that

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$$

$$x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

we know that

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$\text{let } \theta = \sin^{-1} x \quad \text{--- (1)}$$

$$\Rightarrow \sin \theta = x \quad \text{--- (2)}$$

Substitute,

$$\Rightarrow \sin(3 \sin^{-1} x) = 3x - 4x^3$$

$$\Rightarrow \boxed{3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)}$$

H.P.

Q.2 Prove that

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$$

$$x \in \left[\frac{1}{2}, 1\right]$$

we know that

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\text{let } \cos^{-1} x = \theta \quad \checkmark$$

$$\Rightarrow x = \cos \theta \quad \checkmark$$

Substitute,

$$\Rightarrow \cos(3 \cos^{-1} x) = 4x^3 - 3x$$

$$\Rightarrow \boxed{3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)}$$

H.P.Q.3 Prove that $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

$$\boxed{\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)}$$

$$x = \frac{2}{11}, \quad y = \frac{7}{24}$$

$$\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1} \left(\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \cdot \frac{7}{24}} \right) = \tan^{-1}\left(\frac{1}{2}\right)$$

Q.4 Prove that $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$$

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

LHS = $2 \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{7} \right)$

Formula

$$= \tan^{-1} \left(\frac{2 \times \frac{1}{2}}{1 - \frac{1}{4}} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

$$= \tan^{-1} \left(\frac{1}{\left(\frac{3}{4} \right)} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

$$= \tan^{-1} \left(\frac{4}{\frac{3}{\cancel{4}}} \right) + \tan^{-1} \left(\frac{1}{\cancel{7}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{4}{3} + \frac{1}{7}}{\frac{1}{1} - \frac{4}{3} \times \frac{1}{7}} \right) = \tan^{-1} \left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}} \right)$$

$$= \tan^{-1} \left(\frac{31}{17} \right) = \text{RHS}$$

Exercise 2.2 Chapter - 2 (ITF)

Write the following functions in the simplest form:

Q.5 $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$

Substitution: $x = \tan \theta$ $\tan^{-1} x = \theta$

$$= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{\tan^{-1} x}{2}$$

Q.6 $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right), |x| > 1$

Substitution $x = \sec\theta$ $\sec^{-1} x = \theta$

$$= \tan^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\tan\theta) = \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\theta\right)\right)$$

Pattern

$$\textcircled{x^2-1}$$

↓

$$\textcircled{\sec^2\theta-1} = \tan^2\theta$$

$$= \frac{\pi}{2}-\theta = \frac{\pi}{2}-\sec^{-1}x$$

$$\boxed{\text{Q.7}} \quad \tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), \quad 0 < x < \pi$$

$$= \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} \right)$$

$$= \tan^{-1} \sqrt{\tan^2 \frac{x}{2}}$$

$$= \tan^{-1} \left(\tan \frac{x}{2} \right)$$

$$= \frac{x}{2} \quad \checkmark$$

$$\cos(2\theta) = \begin{matrix} 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{matrix}$$

$$\begin{aligned} \cos x &= 2\cos^2 \frac{x}{2} - 1 \\ 1 + \cos x &= 2\cos^2 \frac{x}{2} \\ \cos x &= 1 - 2\sin^2 \frac{x}{2} \\ 2\sin^2 \frac{x}{2} &= 1 - \cos x \end{aligned}$$

$$\boxed{\text{Q.8}} \quad \tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), \quad -\frac{\pi}{4} < x < \frac{3\pi}{4}$$

Nr. & Dr. are divided by cos x

$$= \tan^{-1} \left(\frac{\left(\frac{\cos x - \sin x}{\cos x} \right)}{\left(\frac{\cos x + \sin x}{\cos x} \right)} \right) = \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

$$= \tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \cdot \tan x} \right) = \tan^{-1} \left(\tan \left(\frac{\pi}{4} - x \right) \right)$$

$$= \frac{\pi}{4} - x \quad \checkmark$$

Q.9 $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, $|x| < a$

$\frac{x}{a}$ $\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}$

~~Substitution~~ Nr. & Dr. both are divided by 'a'

$$= \tan^{-1}\left(\frac{\frac{x}{a}}{\frac{\sqrt{a^2-x^2}}{a}}\right) = \tan^{-1}\left(\frac{\frac{x}{a}}{\sqrt{\frac{a^2-x^2}{a^2}}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right)$$

$$1 - (\sin\theta)^2 = \cos^2\theta$$

Substitution

$\frac{x}{a} = \sin\theta$ let $\Rightarrow \sin^{-1}\left(\frac{x}{a}\right) = \theta$

$$= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{\text{Q.10}} \quad \tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), \quad a > 0, \quad -\frac{a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$$

$$x, a \rightsquigarrow \left(\frac{x}{a} \right) \frac{a^3 - 1}{a^3}$$

$$x^3 \rightarrow \frac{x^3}{a^3}$$

Nr. & Dr. are divided by a^3

$$= \tan^{-1} \left(\frac{\frac{3a^2x}{a^3} - \frac{x^3}{a^3}}{\frac{a^3}{a^3} - \frac{3ax^2}{a^3}} \right) = \tan^{-1} \left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2} \right)$$

Substitution

$$\boxed{\frac{x}{a} = \tan \theta}$$

$$\tan^{-1} \left(\frac{x}{a} \right) = \theta$$

$$= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) = \tan^{-1} (\tan 3\theta)$$

$$= 3\theta = 3 \tan^{-1} \left(\frac{x}{a} \right) \checkmark$$

Exercise 2.2

Chapter (2) I.T.F

Find the value of the each of the following

(11) $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$ $\sin^{-1} \frac{1}{2} = \theta \left\{ \frac{\pi}{6} \right.$
 $\frac{1}{2} = \sin \theta \left\{ \theta = \frac{\pi}{6} \right.$

$= \tan^{-1} \left(2 \cos \left(2 \times \frac{\pi}{6} \right) \right)$

$= \tan^{-1} \left(2 \times \frac{1}{2} \right) = \tan^{-1}(1) = \frac{\pi}{4} \checkmark$

Q.12 $\cot \left(\tan^{-1} a + \cot^{-1} a \right) = \cot \left(\frac{\pi}{2} \right) = \cot(90^\circ) = 0$

Q.13 $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$

\downarrow \downarrow

$2 \tan^{-1} x$ $2 \tan^{-1} y$ (By Prop.)

$= \tan \frac{1}{2} \left[2 \tan^{-1} x + 2 \tan^{-1} y \right]$

$= \tan \left(\tan^{-1} x + \tan^{-1} y \right)$

(by $\tan(A+B)$ formula)

$f(f^{-1}(x)) = x$

$= \frac{\tan(\tan^{-1} x) + \tan(\tan^{-1} y)}{1 - \tan(\tan^{-1} x) \cdot \tan(\tan^{-1} y)} = \frac{x + y}{1 - x \cdot y} \checkmark$

Q.14 If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x .

Ans. $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

$$\sin(90^\circ) = 1$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

$$\frac{\pi}{2}$$

$$\therefore \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}(x) = \frac{\pi}{2}$$

Same

$$x = \frac{1}{5}$$

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$$

Property,

Q.15 If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, then find the value of x .

Ans. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$\tan(\text{LHS}) = \tan(\text{RHS})$

$$\Rightarrow \tan\left[\underbrace{\tan^{-1}\left(\frac{x-1}{x-2}\right)}_{\text{A}} + \underbrace{\tan^{-1}\left(\frac{x+1}{x+2}\right)}_{\text{B}}\right] = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x-1}{x-2} + \frac{x+1}{x+2}$$

$$= 1$$

$$\frac{1}{1} = \left(\frac{x-1}{x-2}\right) \cdot \left(\frac{x+1}{x+2}\right)$$

$$\Rightarrow \frac{(x^2 + x - 2) + (x^2 - x - 2)}{\cancel{(x-2)(x+2)}} = 1$$

$$\frac{(x^2 - 4) - (x^2 - 1)}{\cancel{(x-2)(x+2)}}$$

$$\Rightarrow \frac{x^2 + x - 2 + x^2 - x - 2}{x^2 - 4 - x^2 + 1} = 1$$

$$\Rightarrow \frac{2x^2 - 4}{-3} = 1 \Rightarrow 2x^2 - 4 = -3$$

$$\Rightarrow 2x^2 = 4 - 3 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow \boxed{x = \pm \frac{1}{\sqrt{2}}}$$

Exercise 2.2

Q.16 Find the value $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

I-method:

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$$

← Solve →

$$\frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3} \quad \checkmark$$

$$\begin{aligned} \sin \frac{2\pi}{3} &= \sin 120^\circ = \frac{\sqrt{3}}{2} \\ \sin\left(\pi - \frac{\pi}{3}\right) &= \sin \frac{\pi}{3} \end{aligned}$$

II-method:

$$\sin^{-1}\left(\sin \frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$$

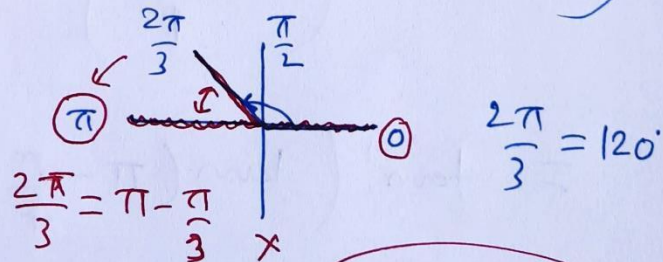
$$\left(\because \sin^{-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right)$$

$$= \sin^{-1}\left(\sin \frac{2\pi}{3}\right)$$

$$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$120 = 180 - 60$$

$(\sin(\pi - \theta) = \sin \theta)$

Q.17 $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

$\left(\frac{n\pi}{2}, \frac{n\pi}{3}, \left(\frac{n\pi}{4}, \frac{n\pi}{6}\right)\right)$

I-method. $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$
↙ Solve. ↘

$= \tan^{-1}(-1)$
↖ IV-quad. ↗

$= -\frac{\pi}{4}$

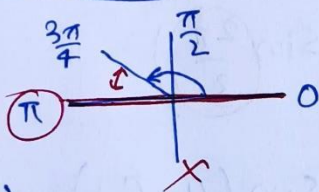
$\tan \frac{3\pi}{4} = \tan 135^\circ$
 $\rightarrow = -1$
 $\tan\left(\pi - \frac{\pi}{4}\right) = -\tan \frac{\pi}{4}$

II-method.

$\tan^{-1}\left(\tan \frac{3\pi}{4}\right) \neq \frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= \tan^{-1}\left(\tan \frac{3\pi}{4}\right)$



$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$\tan(\pi - \theta) = -\tan \theta$

$= \tan^{-1}\left(-\tan \frac{\pi}{4}\right)$

$= -\tan^{-1}\left(\tan \frac{\pi}{4}\right)$

$= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Q.18 $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \tan(A+B)$

\downarrow \downarrow
 $\angle A$ $\angle B$

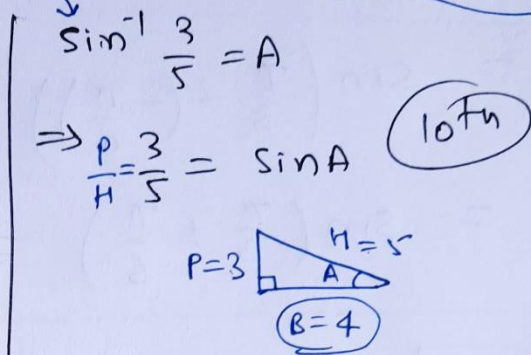
$\sin^{-1}x = \text{angle}$
 $\tan^{-1}x = \text{angle}$
 $\cot^{-1}x = \text{angle}$

Let $\sin^{-1}\frac{3}{5} = A$, $\cot^{-1}\frac{3}{2} = B \Rightarrow \frac{3}{2} = \cot B$

$\Rightarrow \tan(A+B)$
 $= \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$= \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}}$

$\frac{9+8}{4 \times 3} = \frac{17}{6}$



$\tan A = \frac{P}{B} = \frac{3}{4}$

Q.19 $\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$ is equal to (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$

$\cos^{-1}\cos\left(\frac{7\pi}{6}\right) = \frac{7\pi}{6} \notin [0, \pi]$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

$\cos^{-1}x \in [0, \pi]$ Range

$= \cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$= \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{6}\right)\right)$

~~π~~ $\cos(\pi + \theta) = -\cos\theta$

$\Rightarrow \cos^{-1}(-\cos\frac{\pi}{6})$

$= \pi - \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

Q.20 $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (A) $\frac{1}{2}$ (B) $\frac{1}{3}$

$\sin(A-B)$ (C) $\frac{1}{4}$ (D) 1

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
IV-quad.

$= \sin\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Q.21 $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ (A) π (B) $-\frac{\pi}{2}$

angle = $\frac{\pi}{3}$
(I-quad)

angle = $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$
(II-quad)

$= \frac{\pi}{3} - \frac{5\pi}{6}$

$= \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$

Miscellaneous Exercise on Chapter 2

Find the value of the following: →

① $\cos^{-1} \cos \frac{13\pi}{6}$ ② $\tan^{-1} \tan \frac{7\pi}{6}$

① $\cos^{-1} \cos \frac{13\pi}{6} \neq \frac{13\pi}{6}$ $\cos^{-1} x \in [0, \pi]$

(i) method,

$\cos^{-1} \cos \frac{13\pi}{6}$
Solve.

$\frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}$

$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$

$= \cos^{-1} \left[\cos \left(2\pi + \frac{\pi}{6} \right) \right]$

$\cos(2\pi + \theta) = \cos \theta$

$= \cos^{-1} \left(\cos \frac{\pi}{6} \right)$

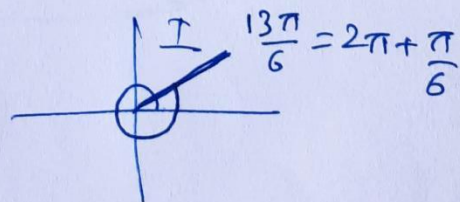
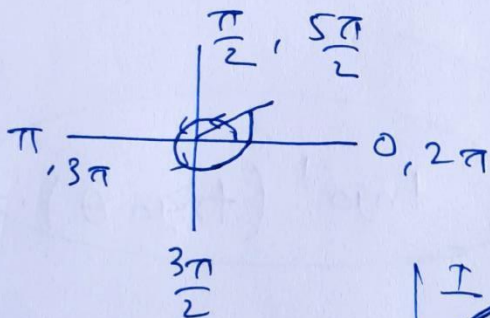
$= \frac{\pi}{6} \in [0, \pi]$

(ii) -method,

$\cos^{-1} \cos \frac{13\pi}{6}$

$= \cos^{-1} \left(\frac{\sqrt{3}}{2} \right)$

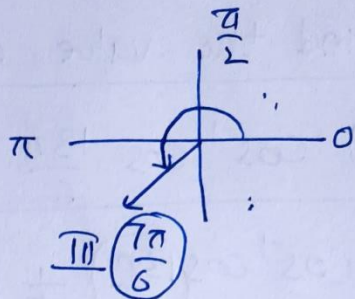
$= \frac{\pi}{6}$



Q.2

$$\tan^{-1}\left(\tan \frac{7\pi}{6}\right) = \frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$$

I method

$$\tan^{-1}\left(\tan \frac{7\pi}{6}\right) = \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{6}$$

II method

$$\tan^{-1}\left(\tan \frac{7\pi}{6}\right)$$

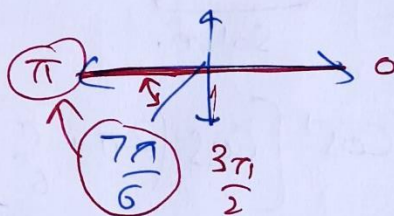
$$= \tan^{-1}\left(\tan\left(\pi + \frac{\pi}{6}\right)\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

✓

$$\tan(\pi + \theta) = \tan \theta$$



$$\frac{3\pi}{2} - \frac{\pi}{3}$$

$$\tan(\pi + \theta) = \tan \theta$$

$\tan^{-1}(\tan \theta) \rightarrow$ Range

Miscellaneous Exercise on Chapter ②

Q3 to Q8

$$\boxed{\text{Q.3}} \quad 2 \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$$

$$\text{let } \sin^{-1} \frac{3}{5} = \theta$$

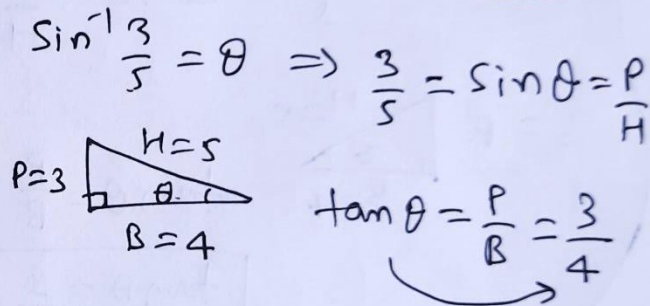
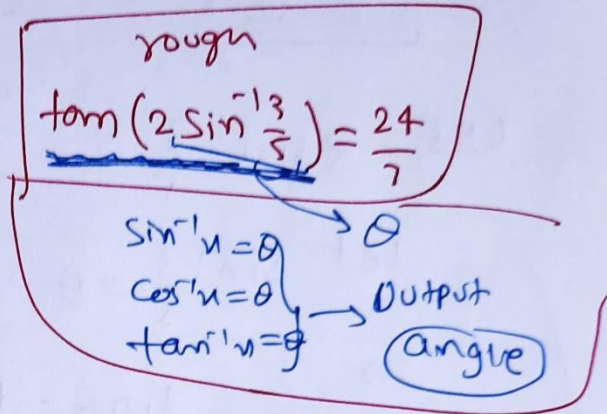
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\Rightarrow \tan 2\theta = \frac{2 \left(\frac{3}{4} \right)}{1 - \left(\frac{3}{4} \right)^2}$$

$$\Rightarrow \tan 2\theta = \frac{3/2}{1 - \frac{9}{16}} = \frac{3/2}{7/16} = \frac{24}{7}$$

$$\Rightarrow \tan 2\theta = \frac{24}{7} \Rightarrow 2\theta = \tan^{-1} \left(\frac{24}{7} \right)$$

$$\Rightarrow \boxed{2 \sin^{-1} \frac{3}{5} = \tan^{-1} \left(\frac{24}{7} \right)}$$



$$\boxed{\text{Q.4}} \quad \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$$

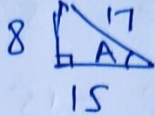
Let A , B
(angles)

$$\tan(A+B) = \frac{77}{36}$$


Now $\tan(A+B)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\tan(A+B) = \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}$$

$$\sin^{-1} \frac{8}{17} = A \Rightarrow \frac{8}{17} = \sin A = \frac{P}{H}$$


$$\tan A = \frac{8}{15} = \frac{P}{B}$$

$$\sin^{-1} \frac{3}{5} = B \Rightarrow \frac{3}{5} = \sin B = \frac{P}{H}$$


$$\tan B = \frac{P}{B_{\text{ax}}} = \frac{3}{4}$$

$$\Rightarrow \tan(A+B) = \frac{32 + 48}{\cancel{15 \times 4}} = \frac{60 - 24}{\cancel{15 \times 4}} = \frac{77}{36}$$

$$\Rightarrow A+B = \tan^{-1} \left(\frac{77}{36} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{8}{17} \right) + \sin^{-1} \left(\frac{3}{5} \right) = \tan^{-1} \left(\frac{77}{36} \right)$$

$$\boxed{\text{Q.5}} \quad \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$$

Let $\underbrace{\cos^{-1} \frac{4}{5}}_A + \underbrace{\cos^{-1} \frac{12}{13}}_B$

$$\cos(A+B) = \frac{33}{65}$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

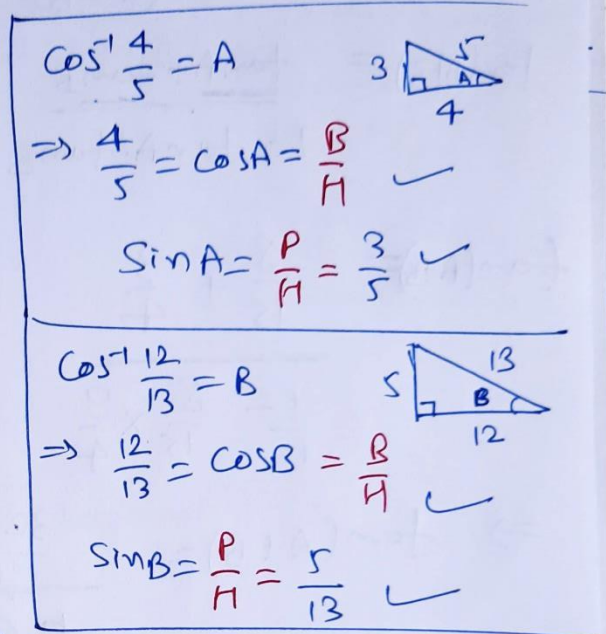
$$\Rightarrow \cos(A+B) = \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$$

$$\Rightarrow \cos(A+B) = \frac{48 - 15}{5 \times 13}$$

$$\Rightarrow \cos(A+B) = \frac{33}{65}$$

$$\Rightarrow A+B = \cos^{-1} \left(\frac{33}{65} \right)$$

$$\Rightarrow \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$$



$$\boxed{\text{Q.6}} \quad \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \left(\frac{56}{65} \right)$$

Let $\underbrace{\cos^{-1} \frac{12}{13}}_A + \underbrace{\sin^{-1} \frac{3}{5}}_B$

$$\sin(A+B) = \frac{56}{65}$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$\Rightarrow \sin(A+B) = \left(\frac{5}{13} \right) \times \left(\frac{4}{5} \right) + \left(\frac{12}{13} \right) \cdot \left(\frac{3}{5} \right)$$

$$\Rightarrow \sin(A+B) = \frac{20}{65} + \frac{36}{65}$$

$$\Rightarrow \sin(A+B) = \frac{56}{65}$$

$$\Rightarrow \textcircled{A+B} = \sin^{-1} \left(\frac{56}{65} \right)$$

$$\Rightarrow \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$$

Q.7

$$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

\downarrow
 \textcircled{A}

\downarrow
B

$$\frac{63}{16} = \tan(A+B)$$

$$\text{Now } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

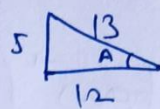
$$\Rightarrow \tan(A+B) = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}}$$

$$\Rightarrow \tan(A+B) = \frac{15 + 48}{\cancel{12 \times 3}} = \frac{36 - 20}{\cancel{12 \times 3}} = \frac{63}{16}$$

$$\Rightarrow A+B = \tan^{-1}\left(\frac{63}{16}\right)$$

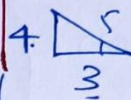
$$\Rightarrow \boxed{\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{63}{16}\right)}$$

$$\sin A = \frac{5}{13} = \frac{P}{H}$$



$$\tan A = \frac{5}{12}$$

$$\cos B = \frac{3}{5} = \frac{B}{H}$$



$$\tan B = \frac{4}{3}$$

$$\boxed{\text{Q.8}} \quad \underbrace{\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)} + \underbrace{\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)} = \frac{\pi}{4}$$

$$\boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

LHS =

$$= \tan^{-1}\left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \cdot \frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \cdot \frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{12}{\cancel{5 \times 7}}}{\frac{34}{\cancel{5 \times 7}}}\right) + \tan^{-1}\left(\frac{\frac{11}{\cancel{3 \times 8}}}{\frac{23}{\cancel{3 \times 8}}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right) =$$

$$= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

Formula

$$= \tan^{-1}\left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{\cancel{325}}{\cancel{17 \times 23}}}{\frac{\cancel{325}}{\cancel{17 \times 23}}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$$

Miscellaneous Exercise on Chapter 2

Q 9, Q 10

Q 11, Q 12

Prove that:

$$\boxed{\text{Q.9}} \quad \tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), \quad x \in [0, 1]$$

$$\text{Put } \boxed{x = \tan^2 \theta}$$

$$\Rightarrow \tan^{-1} \sqrt{\tan^2 \theta} = \frac{1}{2} \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$\Rightarrow \tan^{-1} (\tan \theta) = \frac{1}{2} \cos^{-1} (\cos 2\theta)$$

$$\Rightarrow \theta = \frac{1}{2} (2\theta)$$

$$\Rightarrow \boxed{\theta = \theta}$$

LHS = RHS.

$$\boxed{\text{Q.10}} \quad \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, \quad x \in \left(0, \frac{\pi}{4}\right)$$

~~1/2~~ ~~1/2~~

$$1 + \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2$$

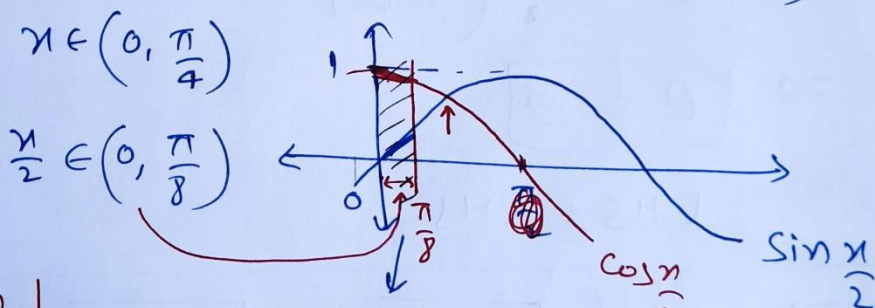
$$1 - \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)^2$$

$$\text{LHS} = \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) \quad x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1} \left(\frac{\sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} + \sqrt{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2}}{\sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} - \sqrt{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2}} \right)$$

$$= \cot^{-1} \left\{ \frac{|\sin \frac{x}{2} + \cos \frac{x}{2}| + |\sin \frac{x}{2} - \cos \frac{x}{2}|}{|\sin \frac{x}{2} + \cos \frac{x}{2}| - |\sin \frac{x}{2} - \cos \frac{x}{2}|} \right\}$$

$\sqrt{x^2} = |x|$



$$\cos \frac{x}{2} > \sin \frac{x}{2}$$

$|\sin \frac{x}{2} - \cos \frac{x}{2}|$
 (Small - Big)

negative \Rightarrow -(negative)

$$-(\sin \frac{x}{2} - \cos \frac{x}{2})$$

$$= \cot^{-1} \left\{ \frac{\cancel{\sin \frac{x}{2}} + \cos \frac{x}{2} - \cancel{\sin \frac{x}{2}} + \cos \frac{x}{2}}{\cancel{\sin \frac{x}{2}} + \cancel{\cos \frac{x}{2}} + \sin \frac{x}{2} - \cancel{\cos \frac{x}{2}}} \right\}$$

$$= \cot^{-1} \left\{ \frac{\cancel{2} \cos \frac{x}{2}}{\cancel{2} \sin \frac{x}{2}} \right\} = \cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$= \frac{x}{2} = \text{RHS.}$$

Q.11 $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$

$-\frac{1}{\sqrt{2}} \leq x \leq 1$ [Hint: $x = \cos 2\theta$]

Solution:

LHS = $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$

$x = \cos 2\theta$

$$= \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right)$$

$\cos 2\theta \begin{cases} \rightarrow 2\cos^2\theta - 1 \\ \rightarrow 1 - 2\sin^2\theta \end{cases}$

$$= \tan^{-1} \left(\frac{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}} \right)$$

$$= \tan^{-1} \left(\frac{(\cos\theta - \sin\theta)/\cos\theta}{(\cos\theta + \sin\theta)/\cos\theta} \right)$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \tan^{-1} \left(\frac{1 - \tan\theta}{1 + \tan\theta} \right)$$

$\frac{\tan A - \tan B}{1 + \tan A \cdot \tan B} = \tan(A-B)$

$$= \tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan\theta}{1 + \tan \frac{\pi}{4} \cdot \tan\theta} \right) = \tan^{-1} \left(\tan \left(\frac{\pi}{4} - \theta \right) \right)$$

$$= \tan^{-1}(\tan(\frac{\pi}{4} - \theta))$$

$$= \frac{\pi}{4} - \theta$$

$$x = \cos 2\theta$$

$$\cos^{-1} x = 2\theta$$

$$\frac{1}{2} \cos^{-1} x = \theta$$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$$

$$= \text{RHS.}$$

Q.12 $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1}(\frac{1}{3}) = \frac{9}{4} \sin^{-1}(\frac{2\sqrt{2}}{3})$ Prove

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4} \sin^{-1}(\frac{1}{3})$$

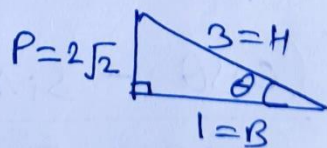
$$= \frac{9}{4} \left[\frac{\pi}{2} - \sin^{-1}(\frac{1}{3}) \right]$$

Property

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$= \frac{9}{4} \left[\underbrace{\cos^{-1}(\frac{1}{3})}_{\theta} \right]$$

$$= \frac{9}{4} \theta$$



$$P^2 + 1 = 9$$

$$P^2 = 8$$

$$P = 2\sqrt{2}$$

$$\sin \theta = \frac{2\sqrt{2}}{3}$$

$$\theta = \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

$$= \frac{9}{4} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

$$= \text{RHS}$$

Miscellaneous Exercise on Chapter (2)

Q 13 to Q 17

Solve the following equations:

Q.13 $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

$x = ?$

$$\Rightarrow \tan\left[2 \tan^{-1}(\cos x)\right] = 2 \operatorname{cosec} x$$

$$\begin{array}{c} \downarrow \\ \tan[2\theta] = \frac{2 \tan(\theta)}{1 - \tan^2 \theta} \\ \swarrow \quad \searrow \\ \text{Formula} \end{array}$$

$$\Rightarrow \frac{2 \tan(\tan^{-1} \cos x)}{1 - [\tan(\tan^{-1} \cos x)]^2} = 2 \operatorname{cosec} x$$

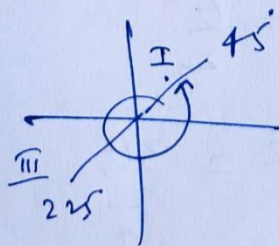
$$\Rightarrow \frac{\cos x}{1 - \cos^2 x} = \frac{1}{\sin x}$$

$$\Rightarrow \frac{\cos x}{\sin^2 x} = \frac{1}{\sin x}$$

$$\Rightarrow \frac{\cos x}{\sin x} = 1$$

$$\Rightarrow \frac{\sin x}{\cos x} = 1$$

$$\Rightarrow \boxed{\tan x = 1} \quad \text{General Solution}$$



$$\begin{array}{l} n \in \mathbb{I} \\ \tan x = \tan \theta \\ x = n\pi + \theta \end{array}$$

$$\tan x = 1 = \tan \frac{\pi}{4}$$

$$\boxed{x = n\pi + \frac{\pi}{4}} \quad n \in \mathbb{Z}$$

ITF Angle
 $\sin^{-1} x \rightarrow \text{angle}$
 $\cos^{-1} x \rightarrow \theta$
 $\tan^{-1} x \rightarrow \theta$

Q.14 $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x$ ($x > 0$)

$x = ?$

I-method

$$2 \tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1} x$$

$$2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}(x)$$

II-method

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x$$

(let $x = \tan \theta$)

$$\Rightarrow \tan^{-1}\left(\frac{1 - \tan \theta}{1 + \tan \theta \cdot 1}\right) = \frac{1}{2} \tan^{-1}(\tan \theta)$$

$$\Rightarrow \tan^{-1}\left(\frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \cdot \tan \theta}\right) = \frac{1}{2}(\theta)$$

$\hookrightarrow \tan(A-B)$

$$\Rightarrow \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \frac{\theta}{2}$$

$$\Rightarrow \frac{\pi}{4} - \theta = \frac{\theta}{2}$$

$$\Rightarrow \frac{\pi}{4} = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{6}}$$

$$\theta = \frac{\pi}{6}$$

Solve $x = ?$

$$x = \tan \theta$$

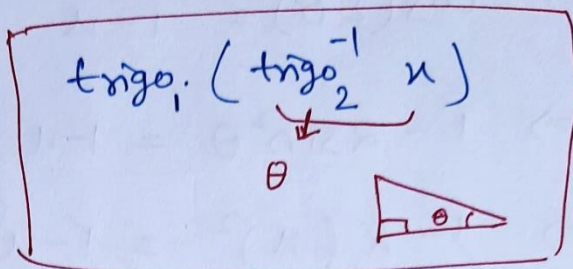
$$x = \tan \frac{\pi}{6}$$

$$x = \frac{1}{\sqrt{3}} \star$$

Q.15 $\sin(\tan^{-1}x)$, $|x| < 1$ is equal to -

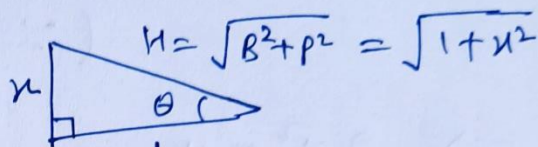
- (A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Ans. $\sin(\tan^{-1}x)$
 \downarrow
 angle = θ (let)



$$\tan^{-1}x = \theta$$

$$\Rightarrow \frac{x}{1} = \tan \theta = \frac{\text{Perp.}}{\text{Base}}$$



Question = $\sin(\tan^{-1}x)$

$$\sin \theta = \frac{P}{H} = \frac{x}{\sqrt{1+x^2}}$$

$$= \sin(\theta)$$

$$= \frac{x}{\sqrt{1+x^2}}$$

Q.16 $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to -

- (A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Solⁿ

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$$

$$\Rightarrow \cos(-2\sin^{-1}x) = 1-x$$

$$\cos(-\theta) = \cos \theta$$

Prop.

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$\sin^{-1}x \rightarrow \theta$

$$\Rightarrow \cos(2 \sin^{-1} x) = 1-x$$

let $\sin^{-1} x = \theta$ (angle) $\Rightarrow x = \sin \theta$

$$\Rightarrow \cos(2\theta) = 1-x$$

$$\Rightarrow 1 - 2\sin^2 \theta = 1-x$$

$$\Rightarrow 1 - 2(x)^2 = 1-x$$

($\because \sin^{-1} x = \theta \Rightarrow x = \sin \theta$)

$$\Rightarrow 2x^2 = x$$

$$\Rightarrow 2x^2 - x = 0$$

$$\Rightarrow x(2x-1) = 0$$

$$x = 0$$

$$2x-1=0 \Rightarrow x = \frac{1}{2}$$

$\cos 2\theta$

- $\cos^2 \theta - \sin^2 \theta$
- $2\cos^2 \theta - 1$
- $1 - 2\sin^2 \theta$ (boxed)
- $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

original equation:

$$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$$

$x=0$ Check

$$\Rightarrow \sin^{-1}(1-0) - 2\sin^{-1}(0) = \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

LHS = RHS

$x=0$ ✓

$x = \frac{1}{2}$ Check

$$\sin^{-1}(1-\frac{1}{2}) - 2\sin^{-1}(\frac{1}{2}) \neq \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{6} - 2\left(\frac{\pi}{6}\right) \neq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{6} \neq \frac{\pi}{2} \quad \times$$

Q.17 $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to -

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $-\frac{3\pi}{4}$

(i)-method:

$\frac{x}{y} = \tan \theta$

(ii) $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

Property

$\tan^{-1}(X) - \tan^{-1}(Y) = \tan^{-1}\left(\frac{X-Y}{1+XY}\right)$

$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y} \cdot \frac{x-y}{x+y}}\right) = \tan^{-1}\left(\frac{\frac{x^2+xy - xy+y^2}{y(x+y)}}{\frac{xy+y^2+x^2-xy}{y \cdot (x+y)}}\right)$

$= \tan^{-1}\left(\frac{x^2+y^2}{y^2+x^2}\right)$

$= \tan^{-1}(1) = \frac{\pi}{4}$