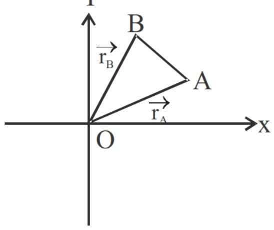
MOTION IN 2D

- When motion of a body/particle is analysed by a moving system, then motion is said to be a relative motion.

- Relative velocity of A w.r. to B is defined as the time rate of change of relative displacement of A w.r. to B, which is given by

$$\vec{V}_{AB} = \frac{d\vec{r}_{AB}}{dt} = \frac{d\overrightarrow{BA}}{dt} = \frac{d}{dt}(\overrightarrow{OA} - \overrightarrow{OB}) = \frac{d\vec{r}_A}{dt} - \frac{d\vec{r}_B}{dt}$$

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B \text{ or } \vec{V}_{BA} = \vec{V}_B - \vec{V}_A$$



$\vec{r}_A$ = position vector of A at time t

$\vec{r}_B$ = position vector of B at time t

Relative velocity is simply the vector difference of two velocities.

- For one dimension $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$

(i) $\longrightarrow A$; $|\vec{V}_{AB}| = |V_A - V_B|$ when motions are along parallel lines

$\longrightarrow B$

(ii) $\longleftarrow B$; $|\vec{V}_{AB}| = V_A + V_B$ when motion are along antiparallel lines.

$\longrightarrow A$

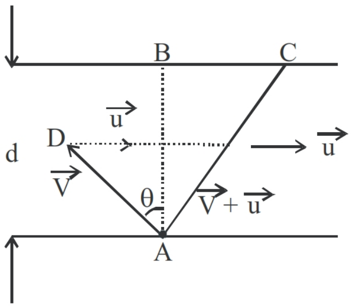
SWIMMER'S PROBLEMS

When boat/swimmer heads in the river to cross from one bank to another. Then motion of boat/swimmer in the direction of resultant of velocity of flow in the river and velocity of boat/swimmer in still water.

$\vec{V}s,g = \vec{V}s,w + \vec{V}w,g$; $\vec{V}s,g$ = velocity of swimmer w.r to ground.

Let $\vec{V}s,w = \vec{V}$ = velocity of swimmer in still water

$\vec{V}w,g = \vec{u}$ velocity of water flow.



Swimmer heads along AD making angle θ with vertical in the direction of upstream so as while it crosses the river it has less drift along the direction of river flow.

- Time to cross the opposite bank = $\dfrac{d}{V\cos\theta}$

  Minimum time to cross the river $= \dfrac{d}{v}$ for which $\theta = 0^0$ i.e. For minimum time to cross the river swimmer should head perpendicular to flow of stream.

- Time to reach just. opposite back (only for v > u)

  $u = v\sin\theta$

  i.e. $\theta = \sin^{-1}\dfrac{u}{v}$ and time to reach opposite bank $= \dfrac{d}{V\sqrt{1-\left(\frac{u}{v}\right)^2}} = \dfrac{d}{\sqrt{V^2-u^2}}$

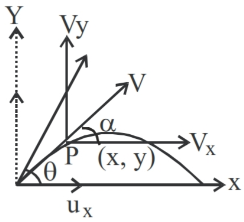
- For v < u then swimmer heads to reach the opposite bank for minimum drift or through shortest path and hence $\dfrac{dBC}{d\theta} = 0$ where BC = $(u - V\sin\theta).\dfrac{d}{V\cos\theta}$

  $\Rightarrow \sin\theta = \dfrac{V}{u}$ or $\theta = \sin^{-1}\left(\dfrac{V}{u}\right)$

  Time to reach the opposite bank through shortest path = $\dfrac{d}{v\sqrt{1-\left(\frac{V}{u}\right)^2}} = \dfrac{du}{v\sqrt{u^2-v^2}}$

PROJECTILE MOTION

An oblique projection of a body from surface of earth the following motion of the body is said to be projectile motion and body itself is called projectile θ is the angle of projection u is velocity of projection. After time t the projectile reaches at P with velocity V.



Then from equation of projectile

$$\vec{a}_x = \frac{d^2\vec{x}}{dt^2} = 0 \quad \text{and} \quad \vec{a}_y = \frac{d^2\vec{y}}{dt^2} = g(-\hat{j})$$

We have $v_x = u_x = u\cos\theta$ and $v_y = u_y - gt = u\sin\theta - gt$

Hence $v = \sqrt{u_x^2 + v_y^2} = \sqrt{u^2 - 2u\sin 2\theta\, gt + g^2t^2}$

and $\alpha = \tan^{-1}\dfrac{v_y}{v_x} = \tan^{-1}\left(\dfrac{u\sin\theta - gt}{u\cos\theta}\right)$

Equation of trajectory or path of projectile is given by $x = u\cos\theta.t$ and $y = u\cos\theta.t - \dfrac{1}{2}gt^2$

Hence we have the equation by eliminating t.

$y = x\tan\theta - \dfrac{1}{2}g\dfrac{x^2}{u^2\cos^2\theta}$. Hence trajectory is a parabolic path.

Range is the horizontal distance from point of projection to the point in the same plane where projectile strikes which is given by

$R = u\cos\theta \times T$; T = time of flight

Since $T = \dfrac{2u\sin\theta}{g}$ $(Sy = 0 = u_y \quad T - \dfrac{1}{2}.gT^2 \quad \Rightarrow u\sin\theta.T - \dfrac{1}{2}gT^2 = 0 \;\Rightarrow\; T = \dfrac{2u\sin\theta}{g}$

$R = \frac{u^2 \sin 2\theta}{g}$. If θ is replaced by $90^0 - \theta$. R does not change.

Hence for given initial velocity R remains the same for two possible values of angle of projections if one is θ then other is $\pi/2 - \theta$.

- Equation of trajectory interms of range $y = x \tan\theta(1 - x/R)$
- Time of ascent = time of descent = $\frac{u \sin\theta}{g} = \frac{u_y}{g}$
- Maximum height – attained by the projectile from plane from where projectile is projected.

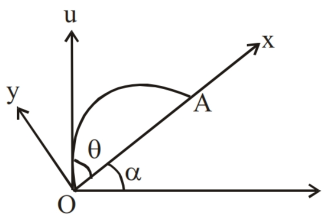
  $H = \frac{u^2 \sin^2\theta}{g} = \frac{u_y^2}{2}$ (At maximum height $v_y^2 = 0 = u_y^2 - 2gH \Rightarrow (u \sin\theta)^2 - 2gH = 0$)
- Along motion of projectile path horizontal velocity remains the same and at hightest point it directs horizontally as no vertical velocity at highest point.
- Every elementary section of projectile path is considered as on curve and the necessary centripetal force required to keep a body on the curve path is pointed along radial direction towards centre of elementary curve path, which is provided by component of weight.
- Time after which the velocity of projectile becomes perpendicular to initial velocity.

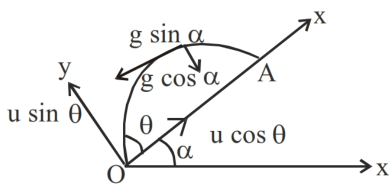
  $\vec{u}.\vec{v} = 0 \Rightarrow (u \cos\theta \hat{i} + u \sin\theta \hat{j}).[u \cos\theta \hat{i} + (u \sin\theta - gt)\hat{j}] = 0$

  $\Rightarrow u^2 - u \sin\theta\, gt = 0$ or $t = \frac{u}{g \sin\theta}$

Projectile Motion on the inclined plane

(i) Projectile Motion up the plane





Taking x-axis along inclined plane and y-axis perepndicular to it at point O.

$\vec{a}_x$ = acceleration along x-axis = $g \sin\alpha(-\hat{i})$

$\vec{a}_y = g \cos\alpha(-\hat{j})$

The time of flight is the time taken for projectile travel from O to A

$\therefore$ From $S_y = u_y t + \frac{1}{2} a y t^2$ for O to A, $S_y = O$

$\therefore \Rightarrow \quad t = \frac{2u \sin\theta}{g \cos\alpha}$

As at t = 0, Projectile is at O.

Time of flight $= \dfrac{2u\sin\theta}{g\cos\alpha}$; Range = OA = R is given

by $S_x = u_x.t + \dfrac{1}{2}a_x t^2$

$$S_x = R = u\cos\theta.\left(\frac{2u\sin\theta}{g\cos\alpha}\right) - \frac{1}{2}g\sin\alpha.\left(\frac{2u\sin\theta}{g\cos\alpha}\right)^2$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta.\cos\alpha - \sin\theta.\sin\alpha\right] \qquad = \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos(\theta+\alpha)\right]$$

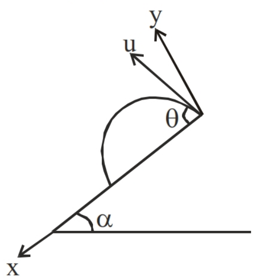
$$R = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha) - \sin\alpha\right]$$

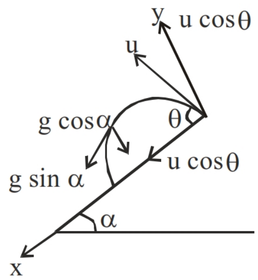
For the maximum-range $\sin(2\theta+\alpha) = 1$; $\theta = 45^0 - \alpha/2$

R_{max} for projection inclined up to plane is $R_{max} = \dfrac{u^2}{g(1+\sin\alpha)}$

(ii) Projectile Motion down the inclined plane

The equation of projectile





$\vec{a}_x = g\sin\alpha\,\hat{i}$ & $\vec{a}_y = g\cos\alpha(-\hat{j})$; Range down the plane $= \dfrac{2u^2\sin\theta\cos(\theta-\alpha)}{g\cos^2\alpha}$

Time of flight $= \dfrac{2u\sin\theta}{g\cos\alpha}$

R_{max} down the plane $= \dfrac{u^2(1+\sin\alpha)}{g\cos^2\alpha} = \dfrac{u^2}{g(1-\sin\alpha)}$

It occurs when direction of projection bisects the angle between the vertical and downward slope of the plane.