Probability

Last Chapter

Strategy

→ Probability class 11 के formulas

→ Probability class 12 के formulas

1. Conditional Probability
2. Multiplication Theorem
3. Baye's Theorem

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4. Probability Distribution of a Random Variable (mean & variance)
5. Bernoulli's Trial & Binomial Distribution

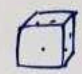
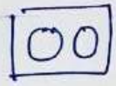
☆ Follow the ORDER

Formula Based Questions	Practical Problems
1 original Reading + 1 Revision	1 original reading + 2 Revision

Class 12 Maths

1 original reading
+
2 revision

Important Formulas/Concepts

1. Random Experiment Throw a Dic 
2. Sample Space $S = \{1, 2, 3, 4, 5, 6\}$
3. Event → $\boxed{\text{what we want}}$
4. Impossible Event $P(\phi) = 0$
5. Sure Event $P(E) = 1$
6. $0 \leq P(E) \leq 1$, $P(S) = 1$
7. Mutually Exclusive Events  $A \cap B = \phi$ (Disjoint Events)
8. Exhaustive Events $\boxed{E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n = S}$
9. $P(A \text{ and } B) = P(A \cap B)$
10. $P(\text{at least one of } A \,\&\, B) = P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
11. For mutually Exclusive Events $P(A \cap B) = 0$
12. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
13. $P(\text{not } A) = P(A') = 1 - P(A)$
14. $P(\text{neither } A \text{ nor } B) = P(A' \cap B')$
15. De Morgan's Law $\boxed{\begin{aligned} A' \cap B' &= (A \cup B)' \\ A' \cup B' &= (A \cap B)' \end{aligned}}$
16. $P(A \text{ but not } B) = P(A \cap B')$

$$\boxed{\text{probability of an Event} = \mathbf{P(E)} = \frac{\text{No. of favourable outcomes}}{\text{total no. of outcomes}}}$$

Conditional Probability (सप्रतिबंध्य प्रायिकता)

Exam

Paper I	Paper II
Pre	Main

Total no. of Candidates 1000 $\xrightarrow{\text{Paper-I}}$ $\boxed{400}$ $\xrightarrow{\text{Paper-II}}$ $\boxed{50}$ $\Rightarrow$

$P(F) = P(\text{any student passes P-I}) = \dfrac{400}{1000} = 0.4$

$P(E) = P(\text{any student passes the complete exam}) = \dfrac{50}{1000} = 0.05$

$P(E|F) = P$ (any student passes the complete exam if it is given that he/she has passed test$_1$) $= \dfrac{50}{400}$

Given

Happened

Condition

Conditional Probability

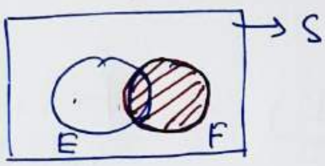
Definition of Conditional Probability.

The conditional probability of event E given that event F has already happened

$$= \boxed{P(E \mid F) = \frac{P(E \cap F)}{P(F)}}$$

Already happened

Proof:



$$P(E \mid F) = \frac{n(E \cap F)/n(S)}{n(F)/n(S)}$$

$$P(E) = \frac{n(E)}{n(S)} \qquad = \frac{P(E \cap F)}{P(F)}$$

Properties of Conditional Probability.

① $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$; $P(F \mid E) = \dfrac{P(F \cap E)}{P(E)}$

② $P(S \mid F) = P(F \mid F) = 1$

③ $P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F) - P((A \cap B) \mid F)$

④ For Mutually Exclusive Events $(A \;\&\; B)$ $A \cap B = \phi$

Disjoint

$$P((A \cap B) \mid F) = 0$$

$$P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F)$$

⑤ $P(E' \mid F) = 1 - P(E \mid F)$

e.g. If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ & $P(A \cup B) = \frac{12}{13}$,

then find (i) $P(A \cap B)$ (ii) $P(A|B)$. (iii) $P(B|A)$.

Ans.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{12}{13} = \frac{7}{13} + \frac{9}{13} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{7}{13} + \frac{9}{13} - \frac{12}{13} = \frac{\overbrace{7+9}^{16}-12}{13} = \frac{4}{13}$$

(i) $\boxed{P(A \cap B) = \frac{4}{13}}$ (ii) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(4/13)}{9/13} = \frac{4}{9}$

(iii) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{4/13}{7/13} = \frac{4}{7}$

e.g. A family has two children. What is the probability that both the children are boys given that atleast one of them is a boy? B = Boy G → Girl

Sample space = S = {BB, BG, GB, GG}

E = event of 'both children are boys' = {BB}

F = 'at least one of them is a boy' = {BB, BG, GB}

$E \cap F = \{BB\}$

$P(E \cap F) = \frac{1}{4}$

$P(F) = \frac{3}{4}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(1/4)}{(3/4)} = \frac{1}{3}$$

S
E
F
E = { }
F = { }
$E \cap F = \{ \}$
P

e.g. A die is thrown twice and the sum of the numbers appearing is observed to be 6. what is the conditional probability that the number 4 has appeared at least once?

F

E

S
E
F

Sample Space $S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$, $n(S) = 36$

E: number '4' appears at least once

F: sum of numbers is 6

S
E
F
$E = \{\ \}$
$F = \{\ \}$
$E \cap F$

$E = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)\} \rightarrow P(E) = \frac{11}{36}$

$F = \{(1,5), (2,4), (3,3), (4,2), (5,1)\} \rightarrow P(F) = \frac{5}{36}$

$E \cap F = \{(2,4), (4,2)\} \rightarrow P(E \cap F) = \frac{2}{36}$

$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{(2/\cancel{36})}{(5/\cancel{36})} = \frac{2}{5}$

? Given

Exercise 13.1 Conditional Probability

Conditional Probability of event E if it is given that event F has already happened $= P(E|F)$

? already happened

$$\boxed{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

$P(E|F) \neq P(F|E)$

Q.1 Given that E & F are events such that $P(E) = 0.6$, $P(F) = 0.3$ and $P(E \cap F) = 0.2$, find $P(E|F)$ & $P(F|E)$.

Ans. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.3} = \frac{2}{3}$

$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{0.2}{0.6} = \frac{2}{6} = \frac{1}{3}$

Q.2 Compute $P(A|B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$.

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.50} = \frac{32}{50} = \frac{16}{25}$

Q.3 If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B|A) = 0.4$, find (i) $P(A \cap B)$ (ii) $P(A|B)$ (iii) $P(A \cup B)$

Ans. (i) $P(B|A) = \frac{P(B \cap A)}{P(A)}$

$\Rightarrow 0.4 = \frac{P(A \cap B)}{0.8}$

$\Rightarrow P(A \cap B) = 0.4 \times 0.8$

$\boxed{P(A \cap B) = 0.32}$

(ii) $P(A|B)$

$= \frac{P(A\cap B)}{P(B)} = \frac{0.32}{0.50}$

$= \frac{0.32}{(1/2)}$

$= 0.32 \times 2 = 0.64$

(iii) $P(A\cup B)$

$P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$= 0.8 + 0.5 - 0.32$

$= 1.30 - 0.32$

$= 0.98$

~~Q.4 Evaluate P(AUB), if 2 P(AUB)~~

Q.4 Evaluate $P(A\cup B)$, if $2.P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$.

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$

$= \frac{5+10-4}{26}$

$= \frac{11}{26}$

$P(B) = \frac{5}{13}$, $P(A) = \frac{5}{26}$

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

$\Rightarrow \frac{2}{5} = \frac{P(A\cap B)}{\left(\frac{5}{13}\right)}$

$\Rightarrow \frac{2}{\cancel{5}} \times \frac{\cancel{5}}{13} = P(A\cap B) = \frac{2}{13}$

Q.5 If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A\cup B) = \frac{7}{11}$, find

(i) $P(A\cap B)$ (ii) $P(A|B)$ (iii) $P(B|A)$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow \frac{7}{11} = \frac{6}{11} + \frac{5}{11} - P(A\cap B)$

$P(A\cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11}$

$\boxed{P(A\cap B) = \frac{4}{11}}$ (i)

(ii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{(4/11)}{(5/11)} = \dfrac{4}{5}$

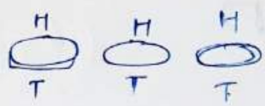
(iii) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{(4/11)}{(6/11)} = \dfrac{\cancel{4}\,2}{\cancel{6}\,3} = \dfrac{2}{3}$ ✓

Determine $P(E|F)$ $\boxed{\text{Q6 to Q9}}$

$\boxed{\text{Q.6}}$ A coin is tossed 3 times, where

(i) E: head on third toss = {HHH, HTH, THH, TTH} ✓

F: heads on first two tosses = {HHH, HHT} ✓



$E \cap F = \{HHH\}$ ✓

Sample Space S = { (HHH), (HHT), (HTH), (THH), (HTT), THT, TTH, TTT }

$\boxed{n(S) = 8}$

$P(F) = \dfrac{2}{8} = \dfrac{1}{4}$ ✓

$P(E \cap F) = \dfrac{1}{8}$ ✓

$P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{(1/\cancel{8})}{(2/\cancel{8})} = \dfrac{1}{2}$

(ii) E: at least two heads (2H, 3H) = {HHH, HHT, HTH, THH}

F: at most two heads (ज्यादा से ज्यादा) (2H, 1H, 0H) = {HHT, HTH, THH, HTT, THT, TTH, TTT}

$E \cap F = \{HHT, HTH, THH\}$

$P(E \cap F) = \dfrac{3}{8}$

$P(F) = \dfrac{7}{8}$

$P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{3/\cancel{8}}{7/\cancel{8}} = \dfrac{3}{7}$

(iii) E: at most two tails (2T, 1T, 0T) = {TTH, THT, HTT, HHT, HTH, THH, HHH}

F = at least one tail (1T, 2T, 3T) = {HHT, HTH, THH, TTH, HTT, THT, TTT}

$E \cap F = \{TTH, THT, HTT, HHT, HTH, THH\}$

$P(E \cap F) = \frac{6}{8}$

$P(F) = \frac{7}{8}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(6/\cancel{8})}{(7/\cancel{8})} = \frac{6}{7}$

$\boxed{Q.7}$ Two coins are tossed once, where

Sample space = S = {HH, HT, TH, TT} $n(S) = 4$

(i) E: tail appears on one coin

F: one coin shows head

$E = \{HT, TH\}$

$F = \{HT, TH\}$

$E \cap F = \{HT, TH\}$

$P(E \cap F) = \frac{2}{4} = \frac{1}{2}$

$P(F) = \frac{2}{4} = \frac{1}{2}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(1/2)}{(1/2)}$

$= 1$

(ii) E: no tail appears

F: no head appears

$E = \{HH\}$

$F = \{TT\}$

$E \cap F = \{\} = \phi$

$P(E \cap F) = P(\phi) = 0$

$P(F) = \frac{1}{4}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{(1/4)}$

$= 0$

Q.8 A die is thrown 3 times,

E: 4 appears on the third toss.

F: 6 and 5 appears respectively on first two tosses.

Sample space:

1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6

$n(S) = 6 \cdot 6 \cdot 6 = 216$

$(\_, \_, 4)$

$$E = \left\{ \begin{array}{llll} (1,1,4) & (1,2,4) & (1,3,4) \ldots & (1,6,4) \\ (2,1,4) & (2,2,4) & (2,3,4) \ldots & (2,6,4) \\ \vdots & \vdots & \vdots & \vdots \\ (6,1,4) & (6,2,4) & (6,3,4) \ldots (6,5,4) & (6,6,4) \end{array} \right\}$$

$$F = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$$

$$E \cap F = \{(6,5,4)\}$$

$\boxed{P(E \cap F) = \frac{1}{216}}$, $\boxed{P(F) = \frac{6}{216}}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(\frac{1}{216}\right)}{\left(\frac{6}{216}\right)} = \frac{1}{6}$$

Q.9 Mother, Father and Son line up at random for a family picture

E : Son on one end F: father in middle

Sample space $S = \{MFS, MSF, FMS, FSM, SFM, SMF\}$

$n(S) = 6$

E: {MFS, FMS, SFM, SMF}

F: {MFS, SFM}

$\rightarrow E \cap F = \{MFS, SFM\}$

$P(E \cap F) = \frac{2}{6}$

$P(F) = \frac{2}{6}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(2/6)}{(2/6)} = 1$$

Q.10 A black and a red dice are rolled

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in 5.

(b) Find the conditional prob. of obtaining the sum 8, given that the ~~black~~ red die resulted in a number less than 4.

$$\text{Sample space} = S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$n(S) = 36$

(a) E : Sum greater than 9

Sum = 10
Sum = 11
Sum = 12

$$E = \left\{ \begin{array}{l} (6,4), (5,5), (4,6), \\ (6,5), (5,6), (6,6) \end{array} \right\}$$

P(E|F)

F: black die results in 5

$$F = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

$$E \cap F = \{(5,5), (5,6)\}$$

$$P(E \cap F) = \frac{2}{36}$$

$$P(F) = \frac{6}{36}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(\frac{2}{36}\right)}{\left(\frac{6}{36}\right)} = \frac{1}{3}$$

(b) E: Sum is 8 $= \{(6,2), (5,3), (4,4), (3,5), (2,6)\}$

F: red die results in number less than 4

$$F = \left\{ \begin{array}{lll} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \\ (4,1) & (4,2) & (4,3) \\ (5,1) & (5,2) & (5,3) \\ (6,1) & (6,2) & (6,3) \end{array} \right\} ; \quad E \cap F = \{(6,2), (5,3)\}$$

$$P(E \cap F) = \frac{2}{36}$$

$$P(F) = \frac{18}{36}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(2/36)}{(18/36)} = \frac{2}{18} = \frac{1}{9}$$

Q11 A fair die is rolled. Consider events $E = \{1, 3, 5\}$, $F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$. Find

$S = \{1, 2, 3, 4, 5\}$

$P(E) = \frac{3}{6}$, $P(F) = \frac{2}{6}$

$P(G) = \frac{4}{6}$

$E \cap F = 3$

$P(E \cap F) = \frac{1}{6}$

(i) $P(E|F)$ & $P(F|E)$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(\frac{1}{6}\right)}{\left(\frac{2}{6}\right)} = \frac{1}{2}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{\left(\frac{1}{6}\right)}{\left(\frac{3}{6}\right)} = \frac{1}{3}$$

(ii) $P(E|G)$ & $P(G|E)$

$E \cap G = \{1, 3, 5\} \cap \{2, 3, 4, 5\}$

$E \cap G = \{3, 5\}$

$P(E \cap G) = \frac{2}{6}$

$$P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{(2/6)}{(4/6)} = \frac{2}{4} = \frac{1}{2}$$

$$P(G|E) = \frac{P(G \cap E)}{P(E)} = \frac{(2/6)}{(3/6)} = \frac{2}{3}$$

(iii) $P((E \cup F)|G)$ & $P((E \cap F)|G)$

$E \cup F = \{1, 3, 5\} \cup \{2, 3\}$
$= \{1, 3, 5, 2\}$

$E \cap F = \{3\}$

$(E \cup F) \cap G = \{1, 3, 5, 2\} \cap \{2, 3, 5, 4\}$
$= \{2, 3, 5\}$

$(E \cap F) \cap G = \{3\}$

$$P((E \cup F)|G) = \frac{P((E \cup F) \cap G)}{P(G)} = \frac{\left(\frac{3}{6}\right)}{\left(\frac{4}{6}\right)} = \frac{3}{4}$$

$$P((E \cap F)|G) = \frac{P((E \cap F) \cap G)}{P(G)} = \frac{(1/6)}{(4/6)} = \frac{1}{4}$$

$\boxed{\text{Exercise 13.1}}$ Conditional Probability

Conditional Probability of E given that F has already happened $= P(E|F) = \dfrac{P(E\cap F)}{P(F)}$

already happened

$\boxed{\text{Q.12}}$ Assume that each born child is equally likely to be a boy or a girl, If a family has two children, what is the cond.ⁿ probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl.

older younger

Sample Space $= S = \{BB, BG, GB, GG\}$

E: both are girls $= \{GG\}$

S
E
F
$E\cap F$

(i) F: the Youngest is a girl $= \{BG, GG\}$

$$P(E|F) = \frac{P(E\cap F)}{P(F)}$$

$$= \frac{(\frac{1}{4})}{(\frac{2}{4})} = \frac{1}{2}$$

$E\cap F = \{GG\}$

$P(E\cap F) = \frac{1}{4}$

$P(F) = \frac{2}{4}$

(ii) F = at least one is a girl

$F = \{BG, GB, GG\}$

$P(E\cap F) = \frac{1}{4}$; $P(F) = \frac{3}{4}$

$E\cap F = \{GG\} \cap \{BG, GB, GG\}$
$= \{GG\}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3} \checkmark$$

$\boxed{Q.13}$

Easy T/F 300	Difficult T/F 200
Easy MCQ 500	Difficult MCQ 400

Total no. of Question = 1400

P($\underbrace{\text{easy question}}_{E}$ given that $\underbrace{\text{it is a MCQ}}_{F}$) = ? = $P(E|F)$

E: selected question is an easy question. (300) Easy T/F + (500) Easy MCQ = 800

F: it is a MCQ. → Easy MCQ (500), Diff. MCQ (400)

$E \cap F$ = Easy MCQ (500)

$$P(E \cap F) = \frac{500}{1400}$$

$$P(F) = \frac{500+400}{1400} = \frac{900}{1400}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(\frac{500}{1400}\right)}{\left(\frac{900}{1400}\right)} = \frac{5}{9}$$

Q. 14 Two Dice

$P($ the sum of numbers on the dice is 4 given that two numbers on two dice are Diff. $)$

E: the sum of numbers on the dice is 4

F: two numbers on two dice are Diff. (Different.)

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

(5)

Sample space $S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$

$n(S) = 36$

E: Sum of numbers on the dice is 4

F: two numbers appearing on two dice are different

$E \cap F = \{(3,1), (1,3)\}$

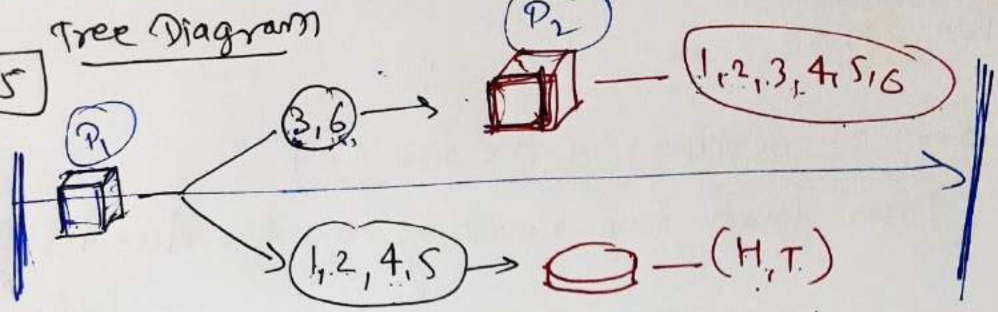
$n(F) = 36 - 6 = 30$

$P(E \cap F) = \frac{2}{36}$; $P(F) = \frac{30}{36}$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{2/\cancel{36}}{30/\cancel{36}} = \frac{1}{15}$$

Q.15

Tree Diagram



$$\boxed{P\Big(\underset{E}{T} \,\Big|\, \underbrace{\text{atleast one die shows a 3}}_{F}\Big)} = ? = P(E|F) = \frac{P(E\cap F)}{P(F)}$$

~~Sample space = S =~~

E : the coin shows a tail = $\{1T, 2T, 4T, 5T\}$

F : atleast one die shows a 3 $= \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (6,3)\}$

$E \cap F = \phi$

Common

$$\boxed{P(E\cap F) = 0}$$

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{0}{P(F)} = 0$$

Q.16 If $P(A) = \frac{1}{2}$, $P(B) = 0$, then $P(A|B)$ is —

(A) 0 (B) $\frac{1}{2}$

(C) not Defined (D) 1

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{0}$$

Q.17 If 'A' and 'B' are events such that $P(A|B) = P(B|A)$, then

(A) $A \subset B$ but $A \neq B$

(B) $A = B$

(C) $A \cap B = \phi$

(D) $P(A) = P(B)$.

$$\Rightarrow \frac{\cancel{P(A \cap B)}}{P(B)} = \frac{\cancel{P(B \cap A)}}{P(A)}$$

$$\Rightarrow \frac{1}{P(B)} = \frac{1}{P(A)}$$

$$\Rightarrow \boxed{P(A) = P(B)}$$

 $\{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$

~~Bef~~ $B = \{2, 5, 6\}$

$P(A) = \frac{3}{6}$, $P(B) = \frac{3}{6}$

Multiplication Theorem on Probability

By Conditional Probability.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \qquad P(F|E) = \frac{P(F \cap E)}{P(E)}$$

$$\Rightarrow \boxed{P(E \cap F) = P(F) \cdot P(E|F)} \qquad \Rightarrow \boxed{P(F \cap E) = P(E) \cdot P(F|E)}$$

P & C

$$P(E \,\&\, F) = P(E \text{ and } F)$$

$$P(EF) = P(E \cap F) = P(E) \cdot P(F|E)$$

$$= P(F) \cdot P(E|F)$$

General - $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|(A \cap B))$

$P(ABC)$

- $P(A \cap B \cap C \cap D) = P(A) \cdot P(B|A) \cdot P(C|(A \cap B)) \cdot P(D|A \cap B \cap C)$

$P(ABCD)$

---

e.g. A box contains 5 black and 6 red balls. Two balls are drawn from the box one after the other without replacement. What is the probability that both drawn balls are black.

Ans: P(both drawn balls are black) $= P(B_1 \cap B_2)$ (and)

$$= P(B_1) \cdot P(B_2|B_1)$$

Event B_1: first ball is black

Event B_2: second ball is black

without replacement

$$P(B_1) = \frac{5}{11}$$

5B + 6R
Total = 11

$$P(B_2 | B_1) = \frac{4}{10}$$

4B + 6R
total = 10

$$P(B_1 \cap B_2) = P(B_1) \cdot P(B_2 | B_1)$$

$$= \frac{5}{11} \cdot \frac{4}{10} = \frac{20}{110} = \frac{2}{11}$$

II - method.

P(both drawn balls are black)

5B, 6R
total = 11

$$= \frac{^5C_2}{^{11}C_2}$$

^nC_r = no. of ways to select r objects out of 'n'

$$= \frac{n!}{(n-r)!\, r!}$$

$$= \frac{\left(\frac{5 \times 4}{\cancel{2 \times 1}}\right)}{\left(\frac{11 \times 10}{\cancel{2 \times 1}}\right)} = \frac{5 \times 4}{11 \times 10} = \frac{20}{110} = \frac{2}{11}$$

e.g. Three Cards are drawn successively, without replacement from a pack of 52 well shuffled Cards. What is the probability that first two Cards are King and the third card is an ace?

$$P(K_1 K_2 A_3) = P(K_1 \cap K_2 \cap A_3)$$

$$= P(K_1) \cdot P(K_2 | K_1) \cdot P(A_3 | K_1 \cap K_2)$$

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{4}{50} = \frac{\cancel{4} \cdot \cancel{3}^{1} \cdot \cancel{4}^{2}}{\cancel{52}_{13} \cdot \cancel{51}_{17} \cdot \cancel{50}_{25}}$$

K = 4	K = 3	K = 2
A = 4	A = 4	A = 4
total = 52	total = 51	total = 50

$$= \frac{2}{5525}$$

Independent Events: Probability of occurrence of one ~~of~~ event is not affected by occurrence of another.

Let E & F are independent Events, then $P(E|F) = P(E)$

$P(F|E) = P(F)$

Multiplication Theorem

$$\boxed{P(E \cap F) = P(E) \cdot P(F|E)}$$

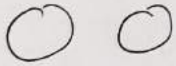
For Independent Events E & F →

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

☆ ← Condition for Independent events.

Dependent Events $\boxed{P(E \cap F) \neq P(E) \cdot P(F)}$

For mutually Exclusive Events



Disjoint

$A \cap B = \phi$

$\boxed{P(A \cap B) = 0}$

For Independent Events

$\updownarrow$

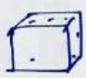
☆ $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

(Independent)

---

e.g. A die is thrown. If **E** is the event 'the number appearing is a multiple of 3' and **F** be the event 'the number appearing is even', then find whether E & F are independent?

Ans.  $S = \{1, 2, 3, 4, 5, 6\}$

$E = \{3, 6\} \longrightarrow P(E) = \frac{2}{6}$

$F = \{2, 4, 6\} \longrightarrow P(F) = \frac{3}{6}$

$E \cap F = \{6\} \longrightarrow \boxed{P(E \cap F) = \frac{1}{6}}$

Condition for Independent Events

$P(E \cap F) = P(E) \cdot P(F)$

$\frac{1}{6} = \frac{1}{6}$

$\boxed{P(E) \cdot P(F) = \frac{2}{6} \times \frac{3}{6} = \frac{\cancel{6}^{\,1}}{\cancel{6} \times 6} = \frac{1}{6}}$

Here: $\therefore \boxed{P(E \cap F) = P(E) \cdot P(F)}$

$\therefore$ E & F $\rightarrow$ Independent events.

e.g. Prove that if E & F are independent events, then E & F' are also independent.

Ans. Given, E & F → Independent $P(F|E) = P(F)$

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

To Prove, E & F' → Independent

$$\boxed{P(E \cap F') = P(E) \cdot P(F')}$$

Proof. $P(E \cap F') = P(E) \cdot P(F'|E)$ (multiplication theorem)

$= P(E) \cdot (1 - P(F|E))$ (Prop. of Conditional Probability)

$= P(E) \cdot (1 - P(F))$

$P(E \cap F') = P(E) \cdot P(F')$

$\therefore$ E & F' are also independent.

Note: If E & F are independent events then

(i) E & F' are independent.

(ii) E' & F are independent.

(iii) E' & F' are independent.

Exercise 13.2 — Multiplication Theorem on Probability

$P(A \& B) = P(A \cap B) = P(B) \cdot P(A|B)$

$\boxed{P(A \cap B) = P(A) \cdot P(B|A)}$

$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

Independent Events A & B

☆ $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

$P(A|B) = P(A)$

$P(B|A) = P(B)$

Q.1 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

For independent events

$P(A \cap B) = P(A) \cdot P(B) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$ ✓

Q.2 Two Cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

I_B & II_B

I_B = First card is black.

II_B = Second card is black.

$P(\text{both cards are black}) = P(I_B \& II_B)$

$= P(I_B \cap II_B) = P(I_B) \cdot P(II_B | I_B)$ — multiplication Theorem

$= \frac{26}{52} \cdot \frac{25}{51}$

$= \frac{\cancel{26}^{1}}{\cancel{52}_{2}} \times \frac{25}{51}$

$= \frac{25}{102}$ ✓

Black = 26
Red = 26
total = 52

Black = 25
Red = 26
total = 51

Q.3 A box of oranges is inspected by examining three randomly selected oranges drawn (without replacement). If all the three oranges are good, the box is approved for sale, otherwise it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

$$P(\text{box approved for sale}) = P(I_G \,\&\, II_G \,\&\, III_G)$$

$$= P(I_G \cap II_G \cap III_G)$$

$$= P(I_G) \cdot P(II_G \mid I_G) \cdot P(III_G \mid I_G \cap II_G)$$

$$= \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13}$$

Good = 12	Good = 11	Good = 10
Bad = 3	Bad = 3	Bad = 3
total = 15	total = 14	total = 13

$$= \frac{\cancel{12}^{\,4}}{\cancel{15}_{\,\cancel{5}}} \times \frac{11}{\cancel{14}_{\,7}} \times \frac{\cancel{10}^{\,2}}{13} = \frac{4 \times 11}{7 \times 13} = \frac{44}{91}$$ ✓

Q.4 A fair coin and an unbiased die are tossed. Let 'A' be the event 'head appears on the coin' and 'B' be the event '3 on the die'. Check whether A & B are independent events or not.

H, T   1, 2, 3, 4, 5, 6

Sample Space $= S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

A: Head on Coin $= \{H1, H2, H3, H4, H5, H6\} = A$

B: '3 on die' $= \{H3, T3\}$; $A \cap B = \{H3\}$

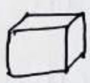
$P(A) = \frac{\cancel{6}\,1}{\cancel{12}\,2} = \frac{1}{2}$ $\qquad$ $P(A \cap B) = \frac{1}{12}$

$P(B) = \frac{\cancel{2}\,1}{\cancel{12}\,6} = \frac{1}{6}$

$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = P(A \cap B)$

$\therefore$ Events A & B are independent events.

$\boxed{Q.5}$ A die marked 1,2,3 in red and 4,5,6 in green is tossed. Let A be the event, 'the number is even', and B be the event, 'the number is red'. Are A & B independent?

Ans.  Sample Space $= S = \{1, 2, 3, 4, 5, 6\}$

A

B

$A = \{2, 4, 6\}$ $\qquad$ $B = \{1, 2, 3\}$

$A \cap B = \{2\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{3}{6} = \frac{1}{2}$, $P(A \cap B) = \frac{1}{6}$

$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \frac{1}{6} = P(A \cap B)$ Not independent

Condition for Independent Events

$P(A \cap B) = P(A) \cdot P(B)$

$\boxed{Q.6}$ Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E & F independent?

Independent

$\boxed{P(E \cap F) = P(E) \cdot P(F)}$

$P(E \cap F) \neq P(E) \cdot P(F)$ Indep.

$\Rightarrow \frac{1}{5} \neq \frac{3}{5} \times \frac{3}{10}$

$\Rightarrow \frac{1}{5} \neq \frac{9}{50}$

Not Independent events

Exercise 13.2

Independent Events : (A & B)

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$
$$\boxed{P(A\cap B) = P(A)\cdot P(B)}$$

Note: If A & B are independent events, then (A & B'), (A' & B), (A' & B') are also independent.

Q.7 Given that the events A & B are such that $P(A) = \frac{1}{2}$, $P(A\cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(i) mutually exclusive. disjoint $A\cap B = \phi$ $\boxed{P(A\cap B) = 0}$

(ii) independent. $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

Ans: $P(A) = \frac{1}{2}$, $P(B) = p$, $P(A\cup B) = \frac{3}{5}$.

$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$ ☆

(i) mutually exclusive, $P(A\cap B) = 0$

$P(A\cup B) = P(A) + P(B)$

$\Rightarrow \frac{3}{5} = \frac{1}{2} + p \Rightarrow \boxed{p = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10} = p}$

(ii) independent, $P(A\cap B) = P(A)\cdot P(B) = \frac{1}{2}\times p = \frac{p}{2}$

$P(A\cup B) = P(A) + P(B) - P(A)\cdot P(B)$

$\Rightarrow \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \Rightarrow \left(\frac{3}{5} - \frac{1}{2}\right) = \frac{p}{2} \Rightarrow \boxed{\frac{1}{10} = \frac{p}{2}} \Rightarrow \boxed{p = \frac{1}{5}}$

Q.8 Let A and B be independent events with $P(A)=0.3$ and $P(B)=0.4$. Find (i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A|B)$ (iv) $P(B|A)$.

Ans. (i) $\boxed{P(A \cap B) = P(A) \cdot P(B)} = (0.3) \times (0.4) = 0.12$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.3 + 0.4 - 0.12 = 0.7 - 0.12 = 0.58$

(iii) $P(A|B) = P(A) = 0.3$

(iv) $P(B|A) = P(B) = 0.4$

Q.9 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$. Find ~~not A~~ $P(\text{not } A \ \& \text{ not } B) = ?$

Independent events.

$P(A \cap B) = P(A) \cdot P(B)$

$\frac{1}{8} = \frac{1}{4} \times \frac{1}{2}$

$\frac{1}{8} = \frac{1}{8}$

$\Rightarrow$ A & B are independent events.

$\Rightarrow$ A' & B' are also independent events.

$\Rightarrow \boxed{P(A' \cap B') = P(A') \cdot P(B')}$

$P(A' \cap B')$

$= \cancel{P(A')}$

$= P(A') \cdot P(B')$

$= (1 - P(A)) \cdot (1 - P(B))$

$= (1 - \frac{1}{4}) \cdot (1 - \frac{1}{2})$

$= \frac{3}{4} \times \frac{1}{2}$

$= \frac{3}{8}$

Q.10 Events A and B are such that $P(A)=\frac{1}{2}$, $P(B)=\frac{7}{12}$ and $P(\text{not } A \text{ or not } B)=\frac{1}{4}$. State whether A and B are independent?

$$\boxed{P(A\cap B)=P(A)\cdot P(B)}$$

$$P(A'\cup B')=\frac{1}{4}$$

$$\Rightarrow P((A\cap B)')=\frac{1}{4}$$

$$\Rightarrow 1-P(A\cap B)=\frac{1}{4}$$

$$\Rightarrow 1-\frac{1}{4}=P(A\cap B)$$

$$\Rightarrow \boxed{P(A\cap B)=\frac{3}{4}} \longleftrightarrow \boxed{P(A)\cdot P(B)=\frac{1}{2}\times\frac{7}{12}=\frac{7}{24}}$$

De morgan's Law

$$\boxed{\begin{aligned} A'\cup B' &= (A\cap B)' \\ A'\cap B' &= (A\cup B)' \end{aligned}}$$

$$\boxed{P(C')=1-P(C)}$$

$\therefore P(A\cap B)\neq P(A)\cdot P(B)$

$\therefore$ A & B are not independent.

Q.11 Given two independent events A and B such that $P(A)=0.3$, $P(B)=0.6$. Find $\longrightarrow$

A & B' = A' & B'

(i) $P(A \text{ and } B) = P(A\cap B) = P(A).P(B) = (0.3)(0.6) = \underline{0.18}$

(ii) $P(A \text{ but not } B) = P(A\cap B') = P(A).P(B') = P(A).(1-P(B))$
$= (0.3)\times(1-0.6) = 0.3\times 0.4 = \underline{0.12}$

(iii) $P(A \text{ or } B) = P(A\cup B) = P(A)+P(B)-P(A\cap B) = 0.3+0.6-0.18$
$= 0.72$

(iv) $P(\text{neither } A \text{ nor } B) = P(A' \,\&\, B') = P(A'\cap B') = P(A').P(B')$
$= (1-P(A)).(1-P(B))$
$= (1-0.3).(1-0.6) = (0.7)\times(0.4) = \underline{\underline{0.28}}$

$\boxed{\text{Exercise 13.2}}$ Multiplication Theorem on Probability

$$P(A \,\&\, B) = P(A \cap B) = P(B) \cdot P(A|B)$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

Independent Events	$P(A \cap B) = P(A) \cdot P(B)$

Note: If A & B are independent events, then
$(A' \,\&\, B)$, $(A \,\&\, B')$, $(A' \,\&\, B')$ $\Rightarrow$ are also independent.

Q. 12 A die is tossed thrice. Find the probability of getting an odd number at least once.

Ans. P(getting an odd number at least once)

(कम से कम एक बार odd no. आये)

||

(Total − (एक बार भी odd no. ना आये))

1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6

$= 1 - P(\text{never getting an odd no.})$

$E_1 \rightarrow$ getting odd no. on 1^{st} throw $\Rightarrow P(E_1) = \frac{3}{6} = \frac{1}{2}$

$E_2 \rightarrow$ ———— 2^{nd} ———— $\Rightarrow P(E_2) = \frac{3}{6} = \frac{1}{2}$

$E_3 \rightarrow$ ———— 3^{rd} ———— $\Rightarrow P(E_3) = \frac{3}{6} = \frac{1}{2}$

$\boxed{E_1 \,\&\, E_2 \,\&\, E_3 \text{ are independent events}}$

P(getting an odd no. at least once)

$= 1 - P(\text{never getting an odd no.})$

$= 1 - P(\underline{E_1'} \ \& \ \underline{E_2'} \ \& \ \underline{E_3'})$ $\qquad (\& \rightarrow \cap)$

$= 1 - P(E_1' \cap E_2' \cap E_3')$

$E_1', E_2', \& E_3' \rightarrow$ Independent

$= 1 - P(E_1') \cdot P(E_2') \cdot P(E_3')$

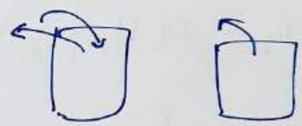
$= 1 - (1 - P(\underline{E_1})) \cdot (1 - P(\underline{E_2})) \cdot (1 - P(\underline{E_3}))$

$= 1 - \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right)$

$= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1 - \frac{1}{8} = \frac{7}{8}$ ✓

Q.13 Two balls are drawn at random with replacement from a box containing 10 Black & 8 Red balls. Find the probability that –

(i) both balls are red.



$P(\text{both balls are red}) = P(I_R \ \& \ II_R)$

I_R : getting 1st ball 'red'
II_R : getting 2nd ball 'red'

$= P(I_R \cap II_R)$

$= P(I_R) \cdot P(II_R \mid I_R)$

$= \frac{8}{18} \times \frac{8}{18}$

10 Black
8 Red
18 total

10 Black
8 Red
18 total

(with replacement)

$= \frac{\cancel{8}^{4}}{\cancel{18}_{9}} \times \frac{\cancel{8}^{4}}{\cancel{18}_{9}}$

$= \frac{16}{81}$ ✓

(II) First ball is black & second is Red.

$I_B \qquad II_R$

$$P(I_B \cap II_R) = P(I_B) \times P(II_R | I_B)$$

$$= \frac{10}{18} \times \frac{8}{18}$$

10 B
8 R
18 total

10 B
8 R
18 total

(with replacement)

$$= \frac{\cancel{10}^{5}}{\cancel{18}_{9}} \times \frac{\cancel{8}^{4}}{\cancel{18}_{9}}$$

$$= \frac{20}{81}$$

(III) one of them is black and other is red.

$$P(I_B \cap II_R) + P(I_R \cap II_B)$$

$$= P(I_B) \cdot P(II_R | I_B) + P(I_R) \cdot P(II_B | I_R)$$

10 B
8 Red
18 total

$$= \frac{10}{18} \times \frac{8}{18} + \frac{8}{18} \times \frac{10}{18}$$

with replacement

$$= \frac{20}{81} + \frac{20}{81} = \frac{40}{81}$$

Q.14 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) problem is solved.

$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{3}$

$$P(\text{problem is solved}) = P(A \text{ or } B) = P(A \cup B)$$

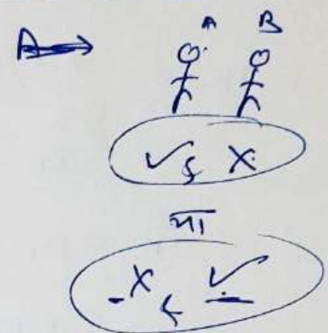
$$= P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - P(A) \cdot P(B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{3+2-1}{6} = \frac{4}{6} = \frac{2}{3}$$

(ii) exactly one of them solves the problem.

$$P(A \cap B') + P(A' \cap B)$$



$$= P(A) \cdot P(B') + P(A') \cdot P(B)$$

$$= \frac{1}{2} \times (1 - P(B)) + (1 - P(A)) \cdot \frac{1}{3}$$

$$= \frac{1}{2} \times \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{2}\right) \cdot \frac{1}{3}$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

A, B → independent

A', B → -"-

A, B' → -"-

$\boxed{\text{Q. 15}}$ One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E & F independent?

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

(i) E : 'the card drawn is spade'

F : 'the card drawn is an ace'

$E \cap F$: the card drawn is (spade & ace)

$$P(E) = \frac{13}{52} \; ; \; P(F) = \frac{4}{52} \; ; \; P(E \cap F) = \frac{1}{52}$$

$$P(E) \cdot P(F) = \frac{\cancel{13}^{1}}{\cancel{52}_{4}} \times \frac{\cancel{4}^{1}}{\cancel{52}_{13}} = \frac{1}{52}$$

E & F → Independent

(ii) E: the card drawn is black

F: the card drawn is a king.

$E \cap F$: the card drawn is ~~a~~ black King.

$P(E) = \frac{26}{52} = \frac{1}{2}$; $P(F) = \frac{\cancel{4}\,1}{\cancel{52}_{13}} = \frac{1}{13}$

$\boxed{P(E \cap F) = \frac{\cancel{2}\,1}{\cancel{52}_{26}} = \frac{1}{26}}$

$\boxed{P(E).P(F) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}}$

$\therefore \boxed{P(E \cap F) = P(E).P(F)}$

$\Rightarrow$ E & F $\rightarrow$ independent

(iii) E: (the card drawn is a King or Queen) = ($K \cup Q$)

F: (the card drawn is a queen or jack) = ($Q \cup J$)

$E \cap F$: the card drawn is a queen

$P(E) = \frac{8}{52} = \frac{2}{13}$; $P(F) = \frac{8}{52} = \frac{2}{13}$;

$\boxed{P(E \cap F) = \frac{4}{52} = \frac{1}{13}} \neq \boxed{P(E).P(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}}$

E & F $\rightarrow$ not independent

(dependent)

Q.16 60% → Hindi News paper

40% → English News paper

20% → Both (H & E).

$P(H) = 60\% = \frac{60}{100} = \frac{3}{5}$

$P(E) = \frac{40}{100} = \frac{2}{5}$

$P(H \cap E) = \frac{20}{100} = \frac{1}{5}$.

(i) P(neither Hindi nor English newspaper)

$= P(H' \cap E')$

$= P((H \cup E)')$

$= 1 - P(H \cup E)$

$= 1 - [P(H) + P(E) - P(H \cap E)]$

$= 1 - \frac{3}{5} - \frac{2}{5} + \frac{1}{5} = \frac{\cancel{5} - \cancel{3} - \cancel{2} + 1}{5} = \frac{1}{5}$

De morgan's law

$H' \cap E' = (H \cup E)'$

$H' \cup E' = (H \cap E)'$

$P(C') = 1 - P(C)$

(ii) P(she reads English if she reads Hindi Newspaper)

given that

Conditional Prob.

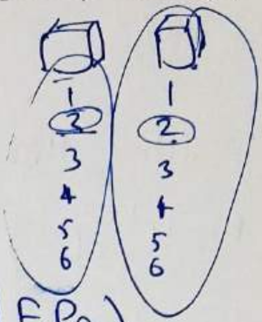
$= P(E|H) = \frac{P(E \cap H)}{P(H)} = \frac{(1/\cancel{5})}{(3/\cancel{5})} = \frac{1}{3}$

(iii) P(she reads Hindi if she reads English News.)

$= P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{(1/\cancel{5})}{(2/\cancel{5})} = \frac{1}{2}$

Q.17 The probability of obtaining an even prime number on each die, when a pair of dice is rolled —

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{12}$ (D) $\frac{1}{36}$



Even Prime No. = 2

$$P(EP_I \cap EP_{II}) = P(EP_I) \cdot P(EP_{II} \mid EP_I)$$

$$= \frac{1}{6} \times \frac{1}{6}$$

$$= \frac{1}{36}$$

$$P(A \cap B) = P(A) \cdot P(B)$$

Q.18 Two events A and B will be independent, if

(A) A & B are mutually exclusive $P(A \cap B) = 0$

(B) $P(A'B') = [1 - P(A)] \cdot [1 - P(B)]$

(C) $P(A) = P(B)$

(D) $P(A) + P(B) = 1$

$$P(A'B') = (1 - P(A)) \cdot (1 - P(B))$$

$$\Rightarrow P(A' \cap B') = 1 - P(B) - P(A) + P(A) \cdot P(B)$$

$$\Rightarrow P((A \cup B)') = 1 - P(B) - P(A) + P(A) \cdot P(B)$$

$$\Rightarrow \cancel{1} - P(A \cup B) = \cancel{1} - P(B) - P(A) + P(A) \cdot P(B)$$

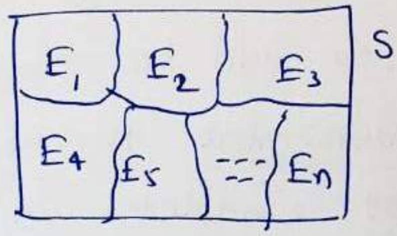
$$\Rightarrow -\cancel{P(A)} - \cancel{P(B)} + P(A \cap B) = -\cancel{P(B)} - \cancel{P(A)} + P(A) \cdot P(B)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

$$P(A' \cap B') = P(A') \cdot P(B')$$

Bayes' Theorem

Partition of a Sample Space



(i) $E_i \cap E_j = \phi$

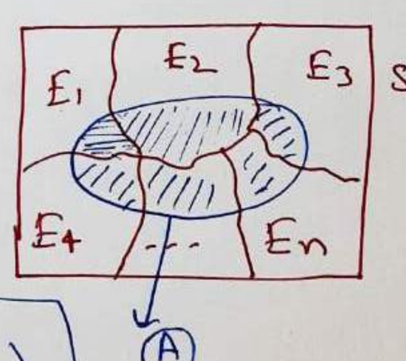
(mutually exclusive events) (Disjoint)

(ii) $E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n = S$

(Exhaustive Events)

(iii) $P(E_i) > 0$ (non zero probabilities)

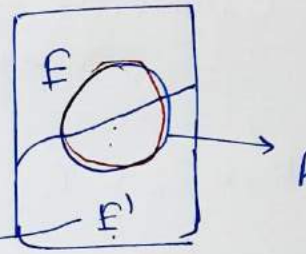
Theorem of total Probability.



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

$$\boxed{P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \cdots + P(E_n) \cdot P(A|E_n)}$$

e.g. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.



E : there will be strike.

E' : there will be no strike

A : job will be completed on time

Q ⇒ $$\boxed{P(A) = P(E).\,P(A|E) + P(E').\,P(A|E')}$$

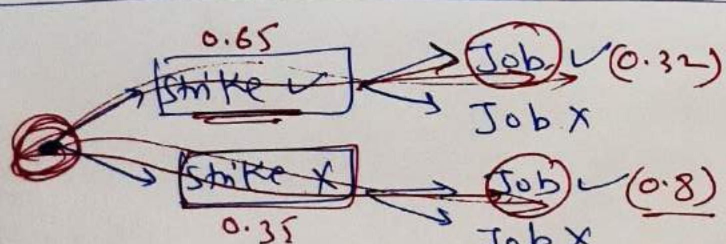
$P(E) = 0.65$

$P(E') = 1 - 0.65 = 0.35$

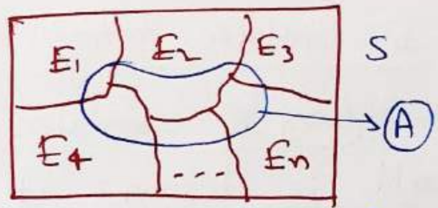
$P(A|E) = 0.32$
(completion of job on time if there is a strike)

$P(A|E') = 0.8$

$$\therefore \boxed{P(A) = 0.65 \times 0.32 + 0.35 \times 0.8 = 0.488}$$



Bayes' Theorem

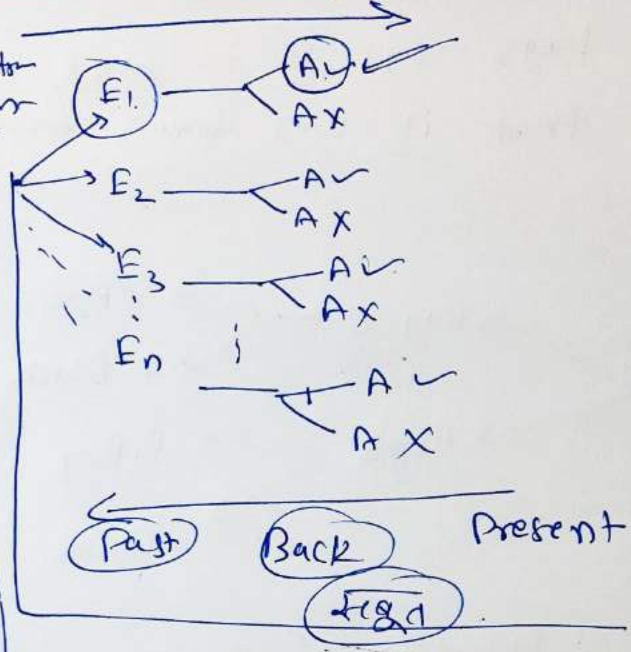


$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)}$$

multiplication theorem

total Probability

$$= \frac{P(E_i) \cdot P(A \mid E_i)}{\left(\begin{array}{l} P(E_1) \cdot P(A \mid E_1) \\ + P(E_2) \cdot P(A \mid E_2) \\ \quad \vdots \\ + P(E_n) \cdot P(A \mid E_n) \end{array} \right)}$$



$$\boxed{P(E_i \mid A) = \frac{P(E_i) \cdot P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j) \cdot P(A \mid E_j)}}$$

Trick

AEPB

A → Arrow Diagram.

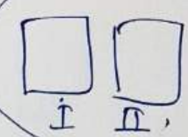
E → Events

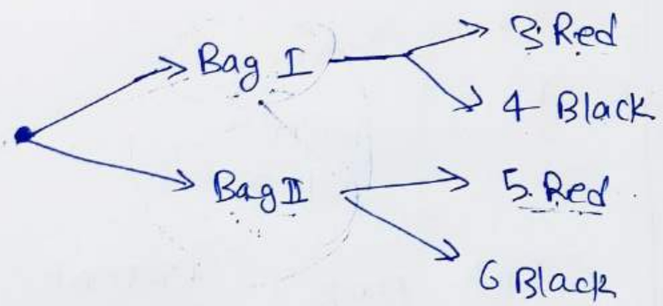
P → Probability

B → Bayes' Theorem

e.g. Bag I contains 3 red and 4 black balls while another Bag II contains 5 Red & 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

$\boxed{\text{AEPB}}$





E_1 : Selection of Bag I

E_2 : Selection of Bag II

A : ball is red

$P(E_1) = \frac{1}{2}$ | $P(A|E_1) = \frac{3}{7}$ ⇒ drawing a red ball from Bag I

$P(E_2) = \frac{1}{2}$ | $P(A|E_2) = \frac{5}{11}$ ⇒ drawing a red ball from Bag II.

Baye's Theorem,

P(~~sect~~ selection of Bag II given that ball is red)

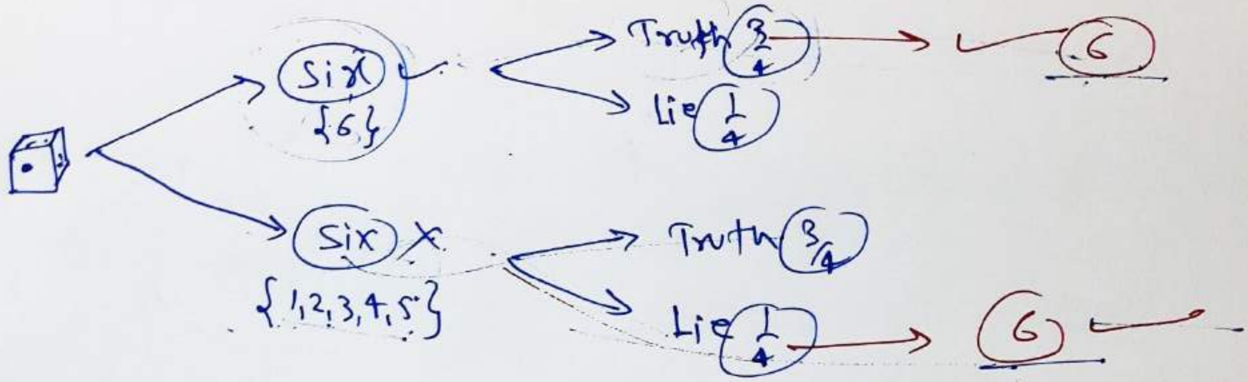
$$= P(E_2|A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\left(\frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{1}{2} \times \frac{5}{11}\right)} = \frac{35}{68}$$

e.g. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

AEPB

Ans.



E_1 : getting 6 on die

E_2 : not getting 6 on die

A : man reports (6)

$P(E_1) = \frac{1}{6}$ | $P(A|E_1) = \frac{3}{4}$ (truth)

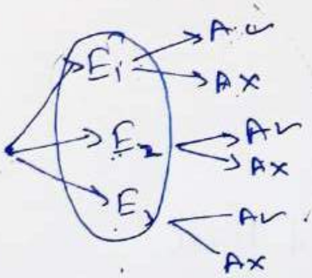
$P(E_2) = \frac{5}{6}$ | $P(A|E_2) = \frac{1}{4}$ (lie)

Bayes' Theorem:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{\frac{3}{24}}{\frac{3}{24}+\frac{5}{24}} = \frac{\frac{3}{\cancel{24}}}{\frac{8}{\cancel{24}}} = \frac{3}{8}$$

$\boxed{\text{Exercise 13.3}}$ Bayes' Theorem



$$\boxed{P(E_1|A) = \frac{P(E_1)\,.\;P(A|E_1)}{\underbrace{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}_{\text{total Probability of A}}}}$$

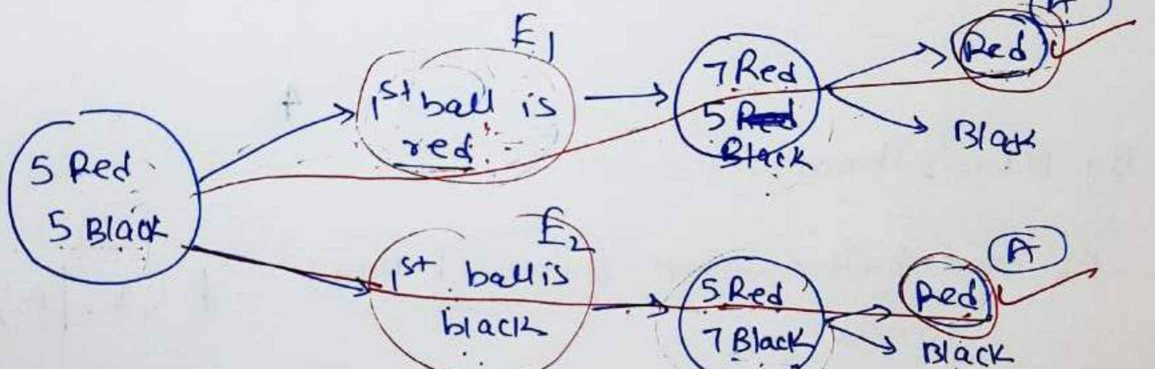
Prob.

$\boxed{\text{A E P B}}$ → Bayes' Theorem

A ↓ Arrow Diagram

E ↓ Events

Q.1



E_1: 1st ball is red $\rightarrow \frac{5}{10} = \frac{1}{2}$ ✓

E_2: 1st ball is black $\rightarrow \frac{5}{10} = \frac{1}{2}$ ✓

A: 2nd ball is red

$\boxed{P(A|E_1) = \frac{7}{12}}$ ✓

$\boxed{P(A|E_2) = 5/12}$ ✓

$P(\text{2nd ball is red}) = P(A) = P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2)$

(total probability)

$\Rightarrow P(A) = \left(\frac{1}{2} \times \frac{7}{12}\right) + \left(\frac{1}{2} \times \frac{5}{12}\right) = \frac{\cancel{12}\; 1}{2 \times \cancel{12}}$

$= \frac{1}{2}$ ✓

Q.2

E_1 Bag I (4 R, 4 B) — Red Ball $\frac{4}{8}$

E_2 Bag II (2 R, 6 B) — Red Ball $= \frac{2}{8}$

Baye's Theorem

AEPB

E_1 : Selecting Bag I

A : getting Red Ball

E_2 : Selecting Bag II

$P(E_1) = \frac{1}{2}$

$P(E_2) = \frac{1}{2}$

$P(A|E_1)$ = getting Red Ball from Bag I

$= \frac{4}{8} = \frac{1}{2}$

$P(A|E_2)$ = getting Red Ball from Bag II

$= \frac{2}{8} = \frac{1}{4}$

By Baye's Theorem:

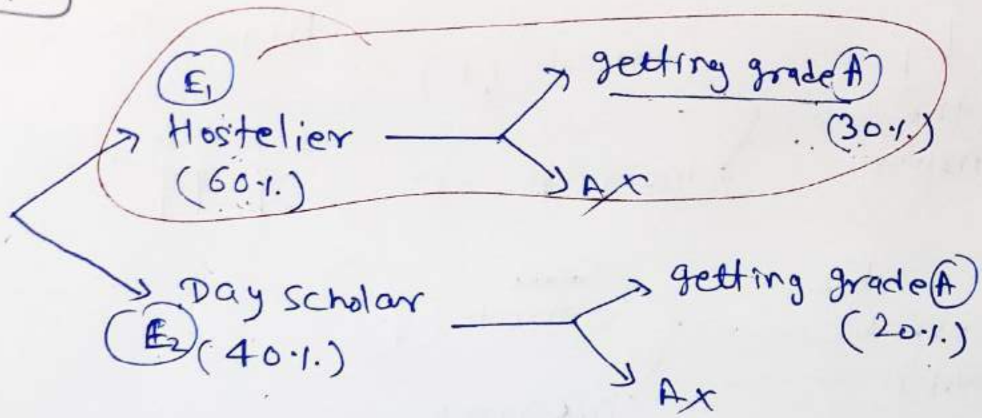
P(red ball (given) came from I Bag) $= P(E_1|A)$

$$\Rightarrow P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$\Rightarrow P(E_1|A) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{2+1}{4_2}}$$

$$= \frac{2}{3}$$

Q.3



Baye's Theorem

A EPB

E_1 : a Student is Hostelier

E_2 : a student is day scholar

A : getting grade 'A'

$P(E_1) = \frac{60}{100} = 0.6$ | $P(A|E_1) = 30\% = \frac{30}{100} = 0.3$

$P(E_2) = \frac{40}{100} = 0.4$ | $P(A|E_2) = 20\% = \frac{20}{100} = 0.2$

~~P(A~~

P(the student is hostelier if it is given that he has 'A' grade)

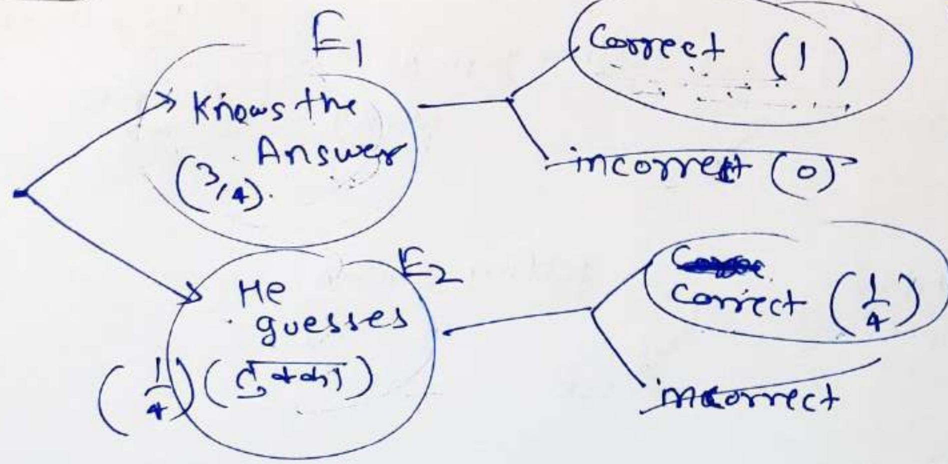
$$\Rightarrow P(E_1|A) = \frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2)}$$

(Bayes' Theorem)

$$= \frac{0.6 \times 0.3}{(0.6 \times 0.3) + (0.4 \times 0.2)} = \frac{0.18}{0.18 + 0.08}$$

$$= \frac{0.18}{0.26} = \frac{18}{26} = \frac{9}{13}$$

Q.4



Baye's Theorem

A E P B

E_1 : He Knows the Answer

E_2: He guesses.

A: he gives Correct Answer.

$P(E_1) = \frac{3}{4}$ $\quad P(A|E_1) = 1$

$P(E_2) = \frac{1}{4}$ $\quad P(A|E_2) = \frac{1}{4}$

P(Student Knows the answer given that he answered it correctly)

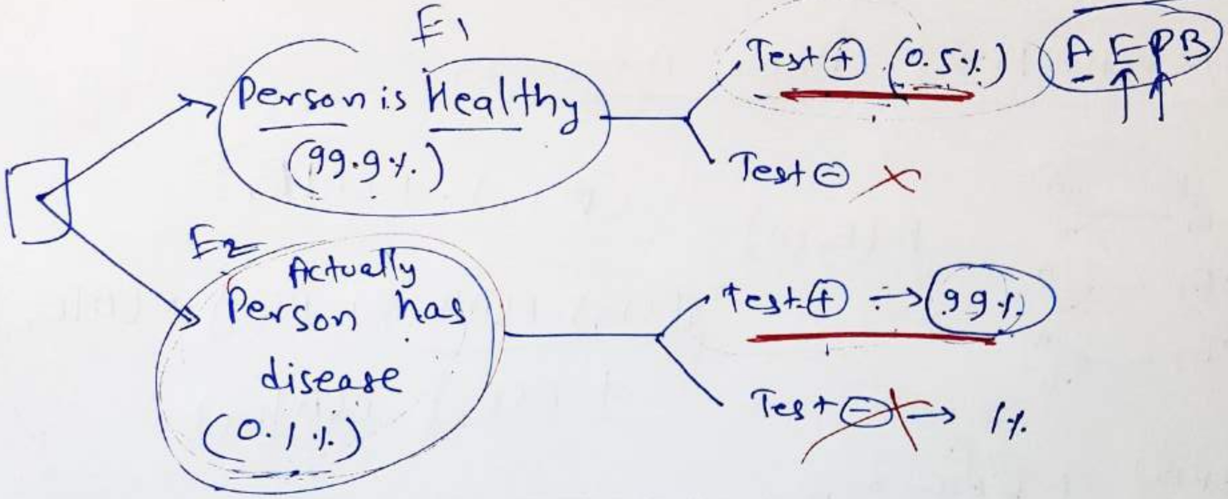
$$= P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

(Bayes' Theorem)

$$= \frac{\left(\frac{3}{4} \times 1\right)}{\left(\frac{3}{4} \times 1\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)} = \frac{\left(3/\cancel{4}\right)}{\left(\frac{12+1}{\cancel{4} \times 4}\right)} = \frac{12}{13}$$

Q. 5

Bayes' Theorem



E_1: A person is healthy.

E_2: A person has disease

A: Test +ve

$P(E_1) = 99.9\% = \frac{99.9}{100} = 0.999$ | $P(A|E_1) = \frac{0.5}{100} = 0.005$

$P(E_2) = 0.1\% = \frac{0.1}{100} = 0.001$ | $P(A|E_2) = \frac{99}{100} = 0.99$

P(a person has disease given that his test result is +ve)

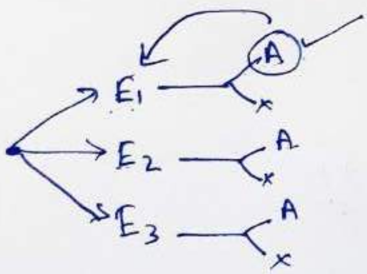
$$P(E_2|A) = \frac{P(E_2) \cdot P(A|E_2)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

(Bayes' theorem)

$$= \left(\frac{0.001 \times 0.99}{0.999 \times 0.005 + 0.001 \times 0.99}\right) = \frac{0.00099}{(0.004995) + (0.00099)}$$

$$= \frac{22}{133}$$

Exercise 13.3 (Bayes' Theorem)



$$P(E_1|A) = \frac{P(E_1)\cdot P(A|E_1)}{\left(P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2) + P(E_3)\cdot P(A|E_3)\right)}$$

Prob.

AEPB A EPB → Bayes'

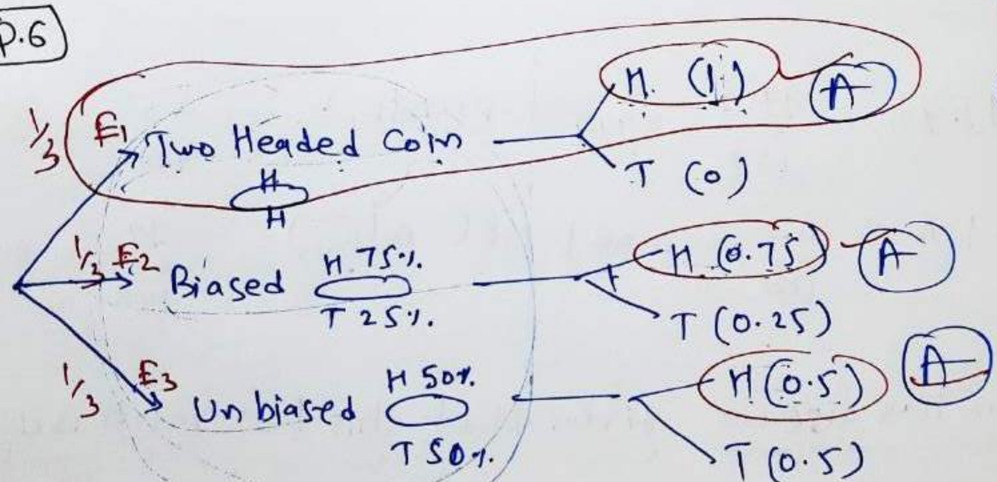
(Arrow Diagram) Event

Q.6

Bayes' Theorem

AEPB

Bay



E_1 : Two headed coin

E_2 : Biased coin

E_3 : unbiased coin

A: the coin shows Head (H)

$P(E_1) = \frac{1}{3} \longrightarrow P(A|E_1) = 1 = 1$

$P(E_2) = \frac{1}{3} \longrightarrow P(A|E_2) = 0.75 = 3/4$

$P(E_3) = \frac{1}{3} \longrightarrow P(A|E_3) = 0.5 = 1/2$

P(selected coin was 2 Headed coin given that the coin shows H)

$$= P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{\left[P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2) + P(E_3) \cdot P(A \mid E_3)\right]}$$

(Bayes' Theorem)

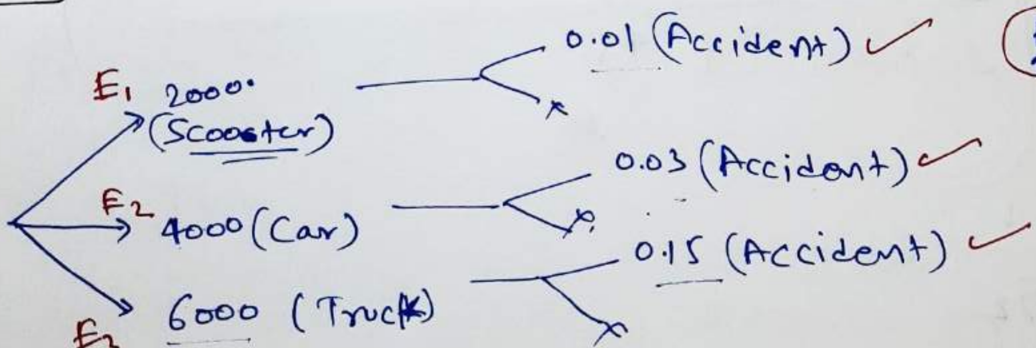
$$= \frac{\cancel{\frac{1}{3}} \times 1}{\left(\cancel{\frac{1}{3}} \times 1\right) + \left(\cancel{\frac{1}{3}} \times \frac{3}{4}\right) + \left(\cancel{\frac{1}{3}} \times \frac{1}{2}\right)} = \frac{1}{\frac{1}{1} + \frac{3}{4} + \frac{1}{2}}$$

$$= \frac{1}{\frac{4+3+2}{4}} = \frac{4}{9}$$

$\boxed{Q.7}$

Bayes' Theorem

AEPB



E_1: he is a scooter Driver

E_2: he is a car driver

E_3: he is a truck driver

A: accident happens

$P(E_1) = \frac{2000}{12000} = \frac{1}{6} \rightarrow P(A \mid E_1) = (0.01)$

$P(E_2) = \frac{4000}{12000} = \frac{1}{3} \rightarrow P(A \mid E_2) = (0.03)$

$P(E_3) = \frac{6000}{12000} = \frac{1}{2} \rightarrow P(A \mid E_3) = (0.15)$

P(he is a scooter driver given that he meets an accident)

$$\Rightarrow P(E_1 | A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + P(E_3) \cdot P(A|E_3)}$$

(Bayes' theorem)

$$= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15}$$

$$= \frac{\left(\frac{1}{6} \times 1\right)}{\frac{1}{6} \times 1 + \frac{1}{3} \times 3 + \frac{1}{2} \times 15}$$

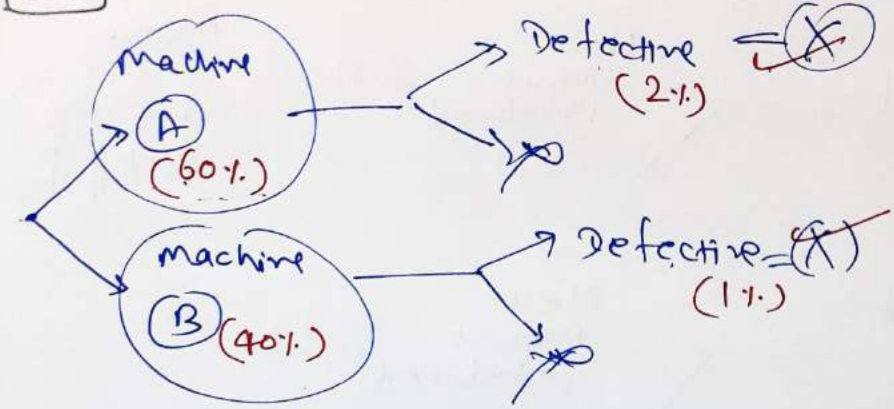
$$= \left[\frac{\left(\cancel{\frac{1}{6}}\right)}{\frac{1 + 6 + 45}{\cancel{6}}} \right]$$

$$= \frac{1}{52}$$

Q. 8

Bayes' Theorem

A E P B



E_1 : item is produced by 'A'

E_2 : item is produced by 'B'

X : item is defective

$P(E_1) = 0.6 \qquad P(X|E_1) = 2\% = \frac{2}{100} = 0.02$

$P(E_2) = 0.4 \qquad P(X|E_2) = 1\% = \frac{1}{100} = 0.01$

P(item produced by machine 'B' given that item is defective)

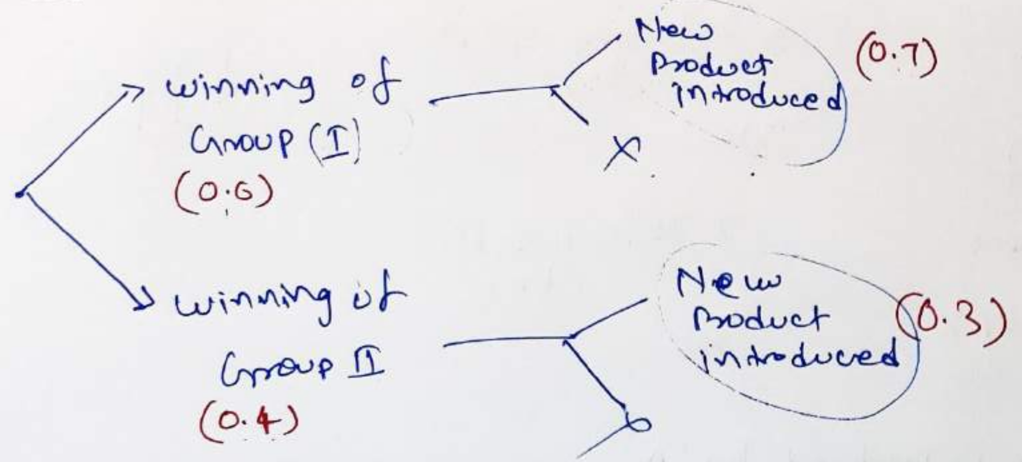
$$P(E_2|X) = \frac{P(E_2) \times P(X|E_2)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2)}$$

(Bayes' Theorem)

$$= \frac{0.4 \times 0.01}{(0.6 \times 0.02) + (0.4 \times 0.01)}$$

$$= \frac{0.004}{0.012 + 0.004} = \frac{4}{12+4} = \frac{4}{16} = \frac{1}{4}$$

Q.9



~~AEBT~~

A E P B

E_1: Winning of Group I

E_2: winning of group II

A: New Product introduced

$P(E_1) = 0.6 \qquad P(A \mid E_1) = 0.7$

$P(E_2) = 0.4 \qquad P(A \mid E_2) = 0.3$

P(new product introduced was by the second group)

$$= P(E_2 \mid A) = \frac{P(E_2) \cdot P(A \mid E_2)}{P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2)}$$

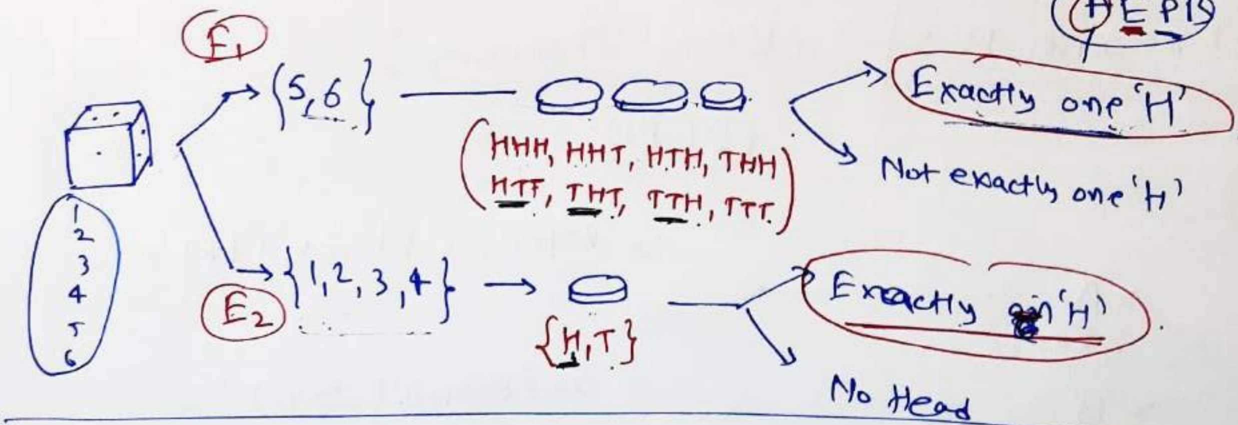
$$= \frac{(0.4 \times 0.3)}{(0.6 \times 0.7) + (0.4 \times 0.3)} = \frac{0.12}{0.42 + 0.12}$$

$$= \frac{0.12}{0.54} = \frac{\cancel{12}^{\,2}}{\cancel{54}_{\,9}} = \frac{2}{9}$$

Q.10

(Bayes' Theorem)

AEPB



E_1: getting 5 or 6 on Die

E_2: getting 1, 2, 3, 4 on Die

A: getting exactly one 'H'

$P(E_1) = \frac{2}{6} = \frac{1}{3} \longrightarrow P(A|E_1) = \frac{3}{8} \leftarrow$ {HTT, THT, TTH}

$P(E_2) = \frac{4}{6} = \frac{2}{3} \longrightarrow P(A|E_2) = \frac{1}{2}$

P(She threw 1, 2, 3, 4 on Die given that She got exactly 1 'H')

$= P(E_2|A) = \dfrac{P(E_2) \cdot P(A|E_2)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$

(Bayes' theorem)

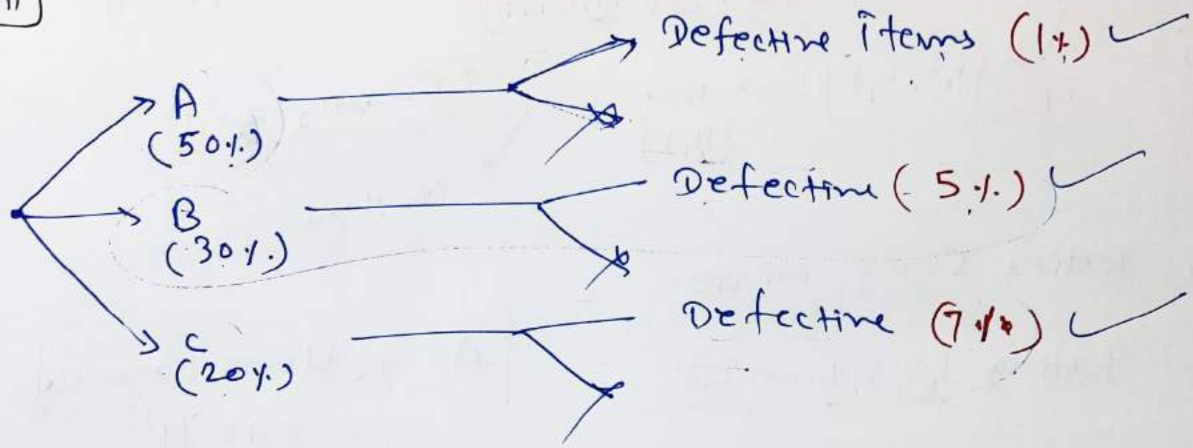
$= \dfrac{\left(\frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}}\right)}{\left(\frac{1}{\cancel{3}} \times \frac{\cancel{3}}{8}\right) + \left(\frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}}\right)} = \dfrac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}}$

$= \dfrac{\left(\frac{1}{\cancel{3}}\right)}{\left(\frac{3+8}{\cancel{24}_{8}}\right)} = \dfrac{8}{11}$

Exercise 13.3 (Bayes' Theorem)

AEPB

Q.11



E_1 : 'A' is on the job

E_2 : 'B' is on the job

E_3 : 'C' ————

F : Defective item is produced.

$P(E_1) = 50\% = \frac{50}{100} = 0.50 \rightarrow P(F|E_1) = 1\% = 0.01$

$P(E_2) = 0.30 \rightarrow P(F|E_2) = 5\% = 0.05$

$P(E_3) = 0.2 \rightarrow P(F|E_3) = 7\% = 0.07$

P (item was produced by A given that item is defective)

$$P(E_1|F) = \frac{P(E_1).P(F|E_1)}{P(E_1).P(F|E_1) + P(E_2).P(F|E_2) + P(E_3).P(F|E_3)}$$

(Bayes' theorem)

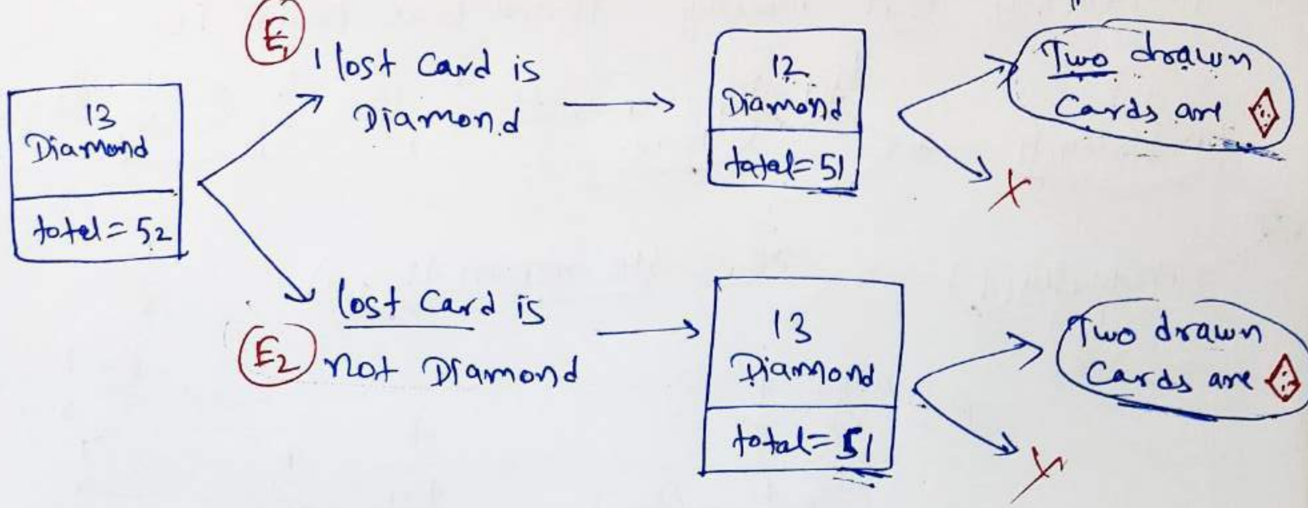
$$= \frac{0.5 \times 0.01}{(0.5 \times 0.01) + (0.3 \times 0.05) + (0.2)(0.07)}$$

$$= 5/34$$

Q.12

Bayes' theorem

AEPB



E_1 : lost card is Diamond

E_2 : lost card is other than Diamond

A : two drawn cards are Diamond

$$P(E_1) = \frac{13}{52} = \frac{1}{4} \rightsquigarrow P(A|E_1) = \frac{^{12}C_2}{^{51}C_2} \quad \frac{12 \diamondsuit}{51 \text{ total}}$$

$$P(E_2) = \frac{39}{52} = \frac{3}{4} \rightsquigarrow P(A|E_2) = \frac{^{13}C_2}{^{51}C_2} = \frac{\left(\frac{13\times 12}{2\times 1}\right)}{\left(\frac{51\times 50}{2\times 1}\right)}$$

P(lost card is Diamond given that two drawn cards are Diamond)

$$= P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

(Bayes' Theorem)

$$= \frac{\frac{1}{4} \times \frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4} \times \frac{^{12}C_2}{^{51}C_2} + \frac{3}{4} \times \frac{^{13}C_2}{^{51}C_2}} = \frac{^{12}C_2}{^{12}C_2 + 3 \times {^{13}C_2}} = \frac{11}{50}$$

Q.13 Probability that 'A' speaks truth is $\frac{4}{5}$. A coin is tossed. 'A' reports that a head appears. The Probability that actually there was head is –

(A) $\frac{4}{5}$ (B) $\frac{1}{2}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$

Actually (H) $\frac{1}{2}$ → $\frac{4}{5}$ Truth → He reports Head.

Actually (T) $\frac{1}{2}$ → $\frac{1}{5}$ Lie → He reports Head

$$P = \frac{\cancel{\frac{1}{2}} \times \frac{4}{\cancel{5}}}{\cancel{\frac{1}{2}} \times \frac{4}{\cancel{5}} + \cancel{\frac{1}{2}} \times \frac{1}{\cancel{5}}} = \frac{4}{4+1} = \frac{4}{5}$$

$\frac{4}{5}$ = $\frac{4}{5}$ = $\frac{1}{5}$

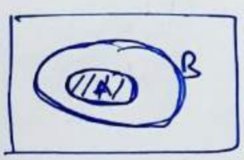
Q.14 If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

~~(A) $P(A|B) = \frac{P(B)}{P(A)}$~~ (B) $P(A|B) < P(A)$

$\boxed{\text{(C) } P(A|B) \geq P(A)}$ (D) None of these

$A \subset B$

Subset



$\boxed{A \cap B = A}$

Common

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \Rightarrow \boxed{P(A|B) = \frac{P(A)}{P(B)}}$$

$$P(B) \leq 1$$

$$\Rightarrow \frac{1}{P(B)} \geq 1$$

$$\Rightarrow \frac{P(A)}{P(B)} \geq P(A)$$

$$\Rightarrow \boxed{P(A|B) \geq P(A)}$$

Random Variable (X)

e.g. Three coins are tossed. Find the probability of 'getting $\underbrace{\text{exactly 2'}}_{\text{Particular value of X}}$ $\underbrace{\text{number of Heads}}_{\text{Random Variable (X)}}$.

e.g. Two dice are thrown. Find the probability of getting sum of numbers, appearing on them, equal to 8.

Random Variable (X) → (Particular value of X)

f(x)

Definition of Random Variable: A random variable is a real valued function whose domain is the sample space of a random experiment. (output = Real No.)

e.g. Consider the random experiment of tossing two coins. If X denotes the number of heads obtained. Is 'X' a random variable? Yes

Sample space $= S = \{HH, HT, TH, TT\}$

$\boxed{X = \text{no. of Heads.}}$

$X(HH) = 2$

$X(HT) = 1 = X(TH)$

$X(TT) = 0.$

$X \rightarrow \boxed{\text{Domain} = S}$

value (output) $= \{0, 1, 2\}$

Range

e.g. A person plays a game of tossing a coin thrice. For each head, he is given ₹100 by the organiser of the game & for each tail, he has to give ₹ 50 to the organiser. Let X denotes the ~~number of~~ amount gained or lost by the person. Show that X is a random variable.

H → + ₹100

T → − ₹50

X = amount gained or lost by the person.

Sample space = S = { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

Random experiment

Domain = S

$X(HHH) = 3 \times 100 = 300$

$X(HHT) = X(HTH) = X(THH) = 2 \times 100 - 1 \times 50 = 150$

$X(HTT) = X(THT) = X(TTH) = 1 \times 100 - 2 \times 50 = 0$

$X(TTT) = (-50) \times 3 = -150$

+ve value → gained (कमाना)

−ve value → lost

Output = Real

X → Real valued fn

Range = {300, 150, 0, −150} ∈ R

Probability Distribution of Random Variable. (X)

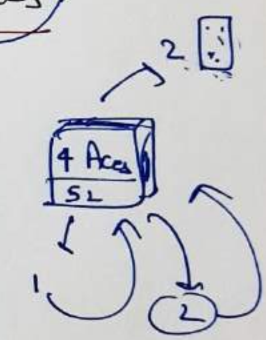
The probability distribution of a random variable X is the system of numbers

X	x_1	x_2	x_3	----	x_n
P(X)	p_1	p_2	p_3	----	p_n

Statistics

where $\boxed{p_i > 0}$, $\boxed{\sum_{i=1}^{n} p_i = 1}$

e.g. Two Cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the no. of aces



Ans. Random Variable X = no. of aces

$X = 0, 1, 2$

Since Card drawing is 'with replacement', therefore two card draws are independent events.

$P(A \cap B) = P(A) \cdot P(B)$

$P(X=0) = P(\text{no. of aces} = 0)$

$= P(\text{non-ace card}_1 \ \& \ \text{non-ace card}_2)$

$= P(\text{non-ace card}_1 \cap \text{non-ace card}_2)$

$= P(\text{non-ace}) \cdot P(\text{non-ace})$

$= \frac{\cancel{48}\,12}{\cancel{52}\,13} \times \frac{\cancel{48}\,12}{\cancel{52}\,13} = \frac{144}{169}$

4 Aces
48 Non aces
52 total

$P(X=1) = P(\text{Ace \& non-ace or non-ace \& Ace})$

$= P(\text{Ace \& non ace}) + P(\text{non-ace \& Ace})$

$= P(\text{Ace}) \cdot P(\text{non-ace}) + P(\text{non ace}) \cdot P(\text{Ace})$

$= \frac{\cancel{4}\,1}{\cancel{52}\,13} \cdot \frac{\cancel{48}\,12}{\cancel{52}\,13} + \frac{48}{52} \times \frac{4}{52}$

$= \frac{12}{169} + \frac{12}{169} = \frac{24}{169}$

$P(X=2) = P(\text{ace and ace})$

$= P(\text{ace} \cap \text{ace})$

$= P(\text{Ace}) \cdot P(\text{Ace})$

$= \frac{\cancel{4}}{\cancel{52}\,13} \times \frac{\cancel{4}}{\cancel{52}\,13} = \frac{1}{169}$

Probability Distribution

X	0	1	2
P(X)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

← No. of aces

verification $P_i > 0$ $\sum P_i = 1$

$\sum P_i = \frac{144}{169} + \frac{24}{169} + \frac{1}{169} = \frac{\cancel{169}\,1}{\cancel{169}} = 1$

Q. Let X denotes the no. of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form, where K is some unknown constant.

$$P(X=x) = \begin{cases} 0.1, & x=0 \\ Kx, & x=1 \text{ or } 2 \\ K(5-x), & x=3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find K.
(ii) What is the probability that you study exactly 2 hrs? At least two hours?

Probability Distribution.

X	0	1	2	3	4
P(X)	0.1	K	2K	2K	K

$\because \sum p_i = 1 \Rightarrow 0.1 + K + 2K + 2K + K = 1$ $\quad 1 - 0.1 = 0.9$

$\Rightarrow 6K = 0.9 \Rightarrow K = \frac{\cancel{0.9}}{\cancel{6}} \frac{\cancel{0.3}}{\cancel{2}} \; 0.15$

(i) $\Rightarrow \boxed{K = 0.15}$

(ii) P(exactly 2 hours study) = $P(X=2) = 2K$
$= 2(0.15)$
$= 0.3$

P(at least 2 hrs study) = $P(X \geq 2)$
$= P(X=2) + P(X=3) + P(X=4)$
$= 2K + 2K + K = 5K = 5(0.15) = 0.75$

Mean, Variance, Standard Deviation of a Random Variable ⇒

Probability Distribution

X	x_1	x_2	x_3	....	x_n
P(X)	P_1	P_2	P_3	....	P_n

- $\boxed{\text{Mean of } X \ (\mu) = \text{Expectation of } X \ (E(X)) = \sum_{i=1}^{n} (p_i x_i)}$

- Variance of X $= Var(X) = \sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

$E(X) = \Sigma p_i x_i$

$E(X^2) = \Sigma p_i x_i^2$

- Standard Deviation (S.D.) $= \sigma_x = \sqrt{Var.(X)}$

e.g. Find the standard deviation of the number obtained on a throw of an unbiased die. X

Ans. Sample space = $\{1, 2, 3, 4, 5, 6\}$

Random Variable = X = Number obtained on a throw.

$X = 1, 2, 3, 4, 5, 6$

$P(X=1) = \frac{1}{6} = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6)$

Probability Distribution

X	1	2	3	4	5	6	← x_i
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	← P_i

$$\text{Mean} = E(X) = \sum_{i=1}^{6} (P_i x_i) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = \frac{1}{6}(21) = \frac{7}{2}$$

$$E(X^2) = \sum (P_i x_i^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{1}{6}(91) = \frac{91}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12}$$

$$Var(X) = \frac{35}{12} = (\sigma_x)^2$$

$$\text{Standard Deviation (SD)} = \sigma_x = \sqrt{Var(X)} = \sqrt{\frac{35}{12}}$$

Exercise 13.4 (Random Variable & its Probability Distribution)

Q.1 State which of the following are not the probability distribution of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P(X)	0.4	0.4	0.2

$P_1 \quad P_2 \quad P_3$

$\boxed{P_i > 0}$ ✓

$\sum P_i = 0.4+0.4+0.2 = 1$ ✓

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

Not $\quad P_4 \not> 0$

(iii)

Y	-1	0	1
P(Y)	0.6	0.1	0.2

$\boxed{P_i > 0}$ Not

X $\sum P_i = 0.6+0.1+0.2 = 0.9 \neq 1$

(iv)

Z	3	2	1	0	-1
P(Z)	0.3	0.2	0.4	0.1	0.05

$\boxed{P_i > 0}$ ✓ Not

X $\sum P_i = 0.3+0.2+0.4+0.1+0.05 = 1+0.05 = 1.05 \neq 1$

Q.2 An urn contains 5 Red and 2 black balls. Two balls are randomly drawn. Let X represents the number of black balls. What are the possible values of X? Is X a random variable?

Ans.

5 Red, 2 Black $\longrightarrow$ 2 Balls

Sample space

$S = \{RR, RB, BR, BB\}$

X = no. of black balls

$X(RR) = 0$

$X(RB) = 1 = X(BR)$

$X(BB) = 2$

① X का Domain = 'S'

② ~~Rand~~. Random experiment

③ Range of $X = \{0, 1, 2\}$

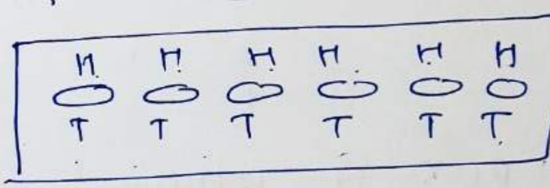
Real no.

$\therefore$ X is a random variable.

Q.3 Let X represents the difference between the number of Heads and the number of tails obtained when a Coin is tossed 6 times. What are possible values of X?

X = difference b/w no. of H & no. of T

$$\boxed{X = |n(H) - n(T)|}$$



$X(HHHHHH) = |6-0| = 6$

$X(HHHHHT) = |5-1| = 4$ ← X(HHHHTH) ---- ----

$X(HHHHTT) = |4-2| = 2$

$X(HHHTTT) = |3-3| = 0$

$X(HHTTTT) = |2-4| = |-2| = 2$

$X(HTTTTT) = |1-5| = |-4| = 4$

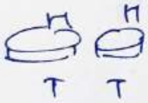
$X(TTTTTT) = |0-6| = |-6| = 6$

Possible values of $X = \{0, 2, 4, 6\}$

Q.4 Find the probability distribution of

(i) number of heads in two tosses of a coin.

Random variable (X) = no. of Heads



$S = \{HH, HT, TH, TT\}$

$X(HH) = 2$

$X(HT) = 1 = X(TH)$

$X(TT) = 0$

$X = \{0, 1, 2\}$ = No. of H ☆

$P(X=0) = P(TT) = \frac{1}{4}$

X	0	1	2
P(X)	1/4	1/2	1/4

$P(X=1) = P(\text{no. of } H = 1)$

$= P(HT \text{ or } TH)$

$= \frac{2}{4} = \frac{1}{2}$

$P(X=2) = P(HH) = \frac{1}{4}$

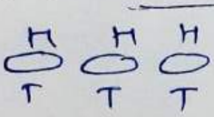
Verification

$P_i > 0$

$\sum P_i = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2} + \frac{1}{2}$

$= 1$

(ii) number of tails in simultaneous tosses of 3 coins.



R.V. = X = no. of tails

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$X = \{0, 1, 2, 3\}$

Prob. Distribution

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

$P(X=0) = P(\text{no. of } T = 0)$

$= P(HHH)$

$= \frac{1}{8}$

$P(X=1) = P(\text{no. of } T = 1)$

$= \frac{3}{8}$

$P(X=2) = P(\text{no. of } T = 2)$

$= 3/8$

$P(X=3) = \frac{1}{8} \leftarrow \{TTT\}$

(iii) Number of heads in four tosses of a coin.

H H H H

T T T T

→ Random variable X = no. of (H)

$$S = \begin{Bmatrix} HHHH, HHHT, HHTH, HTHH, THHH, \\ HHTT, HTHT, THHT, HTTH, THTH, TTHH \\ HTTT, THTT, TTHT, TTTH, TTTT \end{Bmatrix}$$

$\boxed{n(S) = 16}$

$$X = \{4, 3, 2, 1, 0\}$$

Probability Distribution

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$P\cancel{(X)}$

$$P(X=0) = P(\text{no. of } H = 0)$$
$$= \frac{1}{16} \leftarrow \{TTTT\}$$

$$P(X=1) = P(\text{no. of } H = 1)$$
$$= \frac{4}{16} = \frac{1}{4}$$

$$P(X=2) = P(\text{no. of } H = 2)$$
$$= \frac{6}{16} = \frac{3}{8}$$

$$P(X=3) = \frac{4}{16} = \frac{1}{4}$$

$$P(X=4) = P(\text{no. of } H = 4)$$
$$= \frac{1}{16} \leftarrow \{HHHH\}$$

Q.5 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as —

(i) number greater than 4

Random variable X = no. of Successes.

X = no. of (number greater than 4)

$= \{0, 1, 2\}$

0	1	2
(1,1)	(1,5)	(5,5)
(1,2)	(1,6)	(5,6)
(3,2)	(6,3)	(6,5)
⋮	(6,4)	(6,6)
	⋮	

Sample space for two tosses of a Die

$X=0$

1,1	1,2	1,3	1,4	1,5	1,6
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$\rightarrow X=1$

$X=1$

$X=2$

$$P(X=0) = \frac{\cancel{16}^{4}}{\cancel{36}_{9}} = \frac{4}{9}$$

$$P(X=1) = \frac{16}{36} = \frac{4}{9}$$

$$P(X=2) = \frac{4}{36} = \frac{1}{9}$$

Probability Distribution

X	0	1	2
$P(X)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

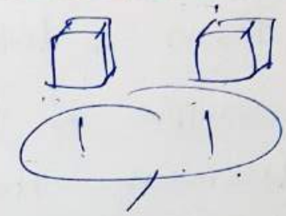
Q.5 (ii) Six appears on at least one die. = Success

Random variable = X = no. of successes

$\boxed{X = \text{no. of times (6 appears on at least one Die)}}$

Sample space for two tosses of a Die

$X = \{0, 1\}$



11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$\boxed{X = 0}$

$X = 1$

at least ~~on~~ one die

(on one die / on two Die)

$P(X=0) = \frac{25}{36}$

$P(X=1) = \frac{11}{36}$

Probability Distribution

X	0	1
P(X)	$\frac{25}{36}$	$\frac{11}{36}$

$\boxed{\text{Exercise 13.4}}$ (Probability Distribution of a Random Variable)

$\boxed{\text{Q.6}}$ From a lot of 30 bulbs which include 6 defective, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

$P(A \cap B) = P(A) \cdot P(B)$

6 Defective
24 Non Def:
30 total

4 bulbs Draw

with replacement ↓ Independent Draws

Random Variable (X) = no. of defective bulbs

$X = \{0, 1, 2, 3, 4\}$

N = non defective
D = Defective

Probability Distribution

X	0	1	2	3	4
$P(X)$	$\frac{256}{625}$	$\frac{64}{625}$	$\frac{16}{625}$	$\frac{4}{625}$	$\frac{1}{625}$

$P(X=0) = P(NNNN)$

$= P(N \cap N \cap N \cap N)$

$= P(N) \cdot P(N) \cdot P(N) \cdot P(N)$

$= \frac{\cancel{24}^{4}}{\cancel{30}_{5}} \cdot \frac{24}{30} \cdot \frac{24}{30} \cdot \frac{24}{30}$

$= \frac{4^4}{5^4} = \frac{256}{625}$

$P(X=1) = P(D.NNN)$

$= P(D) \cdot P(N) \cdot P(N) \cdot P(N)$

$= \frac{\cancel{6}^{1}}{\cancel{30}_{5}} \cdot \frac{\cancel{24}^{4}}{\cancel{30}_{5}} \cdot \frac{\cancel{24}^{4}}{\cancel{30}_{5}} \cdot \frac{\cancel{24}^{4}}{\cancel{30}_{5}}$

$= \frac{64}{625}$

$P(X=2) = P(DDNN)$

$= \frac{6}{30} \cdot \frac{6}{30} \cdot \frac{24}{30} \cdot \frac{24}{30}$

$= \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{4}{5} \cdot \frac{4}{5}$

$= \frac{16}{625}$

$P(X=3) = P(DDDN)$

$= \frac{6}{30} \cdot \frac{6}{30} \cdot \frac{6}{30} \cdot \frac{24}{30}$

$= \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{4}{5}$

$= \frac{4}{625}$

$P(X=4)$

$= P(DDDD)$

$= \frac{6}{30} \cdot \frac{6}{30} \cdot \frac{6}{30} \cdot \frac{6}{30}$

$= \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5}$

$= \frac{1}{625}$

$\boxed{Q.7}$ A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

ATQ. $\boxed{P(H) = 3 \times P(T)}$

H H
T T

$S = \{HH, HT, TH, TT\}$

$\because P(H) + P(T) = 1$

$\Rightarrow 3P(T) + P(T) = 1$

$\Rightarrow 4P(T) = 1$

$\Rightarrow \boxed{P(T) = \frac{1}{4}}$

$\boxed{P(H) = \frac{3}{4}}$

Random Variable X = no. of Tails.

$X = \{0, 1, 2\}$

Independent tossing

$P(A \cap B) = P(A) \cdot P(B)$

Probability Distribution.

X	0	1	2
P(X)	9/16	3/8	1/16

$P(X=0) = P(HH) = P(H \& H)$

$= P(H \cap H)$

$= P(H) \cdot P(H)$

$= \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

$P(X=1) = P(HT) + P(TH)$

$= P(H) \cdot P(T) + P(T) \cdot P(H)$

$= \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{6}{16} = \frac{3}{8}$

$P(X=2) = P(TT)$

$= P(T) \cdot P(T)$

$= \frac{1}{4} \cdot \frac{1}{4}$

$= \frac{1}{16}$

Q.8 A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Determine (i) K (ii) $P(X<3)$ (iii) $P(X>6)$ (iv) $P(0<X<3)$

$P_i > 0$

$\Sigma P_i = 1$ → $0+K+2K+2K+3K+K^2+2K^2+7K^2+K=1$

$\Rightarrow 10K^2+9K-1=0$

$\Rightarrow 10K^2+10K-K-1=0$

$\Rightarrow 10K(K+1)-1(K+1)=0$

$\Rightarrow (K+1)\cdot(10K-1)=0$

$K=-1$ ✗ , $K=\frac{1}{10}$ ✓ (i)

(ii) $P(X<3)$

$= P(X=2)+P(X=1)$

$= 2K+K = 3K$

$= 3\left(\frac{1}{10}\right) = \frac{3}{10} = 0.3$

(iii) $P(X>6)$

$= P(X=7)$

$= 7K^2+K = 7\left(\frac{1}{10}\right)^2+\frac{1}{10}$

$= \frac{7}{100}+\frac{1}{10} = \frac{7+10}{100} = \frac{17}{100}$

(iv) $P(0<X<3)$

$= P(X=1)+P(X=2)$

$= K+2K = 3K = \frac{3}{10}$

Q.9 The random variable X has a probability distribution $P(x)$ of the following form, where k is some number:

$$P(x) = \begin{cases} K, & \text{if } x=0 \\ 2K, & \text{if } x=1 \\ 3K, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Determine the value of K

(ii) Find $P(X<2)$, $P(X \le 2)$, $P(X) \geq 2$.

x	0	1	2
$P(x)$	K	$2K$	$3K$

$P_i > 0$

$\boxed{\Sigma P_i = 1}$

$\Rightarrow K + 2K + 3K = 1$

$\Rightarrow 6K = 1$

$\Rightarrow K = \frac{1}{6}$ (i)

(ii) $P(X<2)$

$= P(X=0) + P(X=1)$

$= K + 2K = 3K = 3\left(\frac{1}{6}\right) = \frac{1}{2}$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= K + 2K + 3K = 6K = 1$

$P(X \geq 2) = P(X=2)$

$= 3K = 3\left(\frac{1}{6}\right) = \frac{1}{2}$

Exercise 13.4

Probability Distribution of a Random Variable

X	x_1	x_2	x_3	...	x_n
$P(x)$	P_1	P_2	P_3	--	P_n

$\sum P_i = 1$

Mean of X (μ) = Expection of X $E(x)$ $= \sum_{i=1}^{n} p_i x_i$

$\boxed{Q.10}$ Find the mean number of heads in three tosses of a fair coin.

Random variable $(X) =$ no. of heads.

$X = \{0, 1, 2, 3\}$

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

~~P(x=0)~~

$P_1 = P(x=0) = P(TTT) = \frac{1}{8}$

$P_2 = P(X=1) = \frac{3}{8}$

$P_3 = P(X=2) = \frac{3}{8}$

$P_4 = P(X=3) = \frac{1}{8}$

Probability Distribution

x	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Mean of $X = \sum_{i=} P_i x_i$

$= P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4$

$= \left(\frac{1}{8}\cdot 0\right) + \left(\frac{3}{8}\cdot 1\right) + \left(\frac{3}{8}\cdot 2\right) + \left(\frac{1}{8}\cdot 3\right)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$= \frac{12}{8} = \frac{3}{2} = 1.5$

Q.11 Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

X = number of sixes.

$= \{0, 1, 2\}$

$P(X=0) = \frac{25}{36}$

$P(X=1) = \frac{10}{36}$

$P(X=2) = \frac{1}{36}$

Sample Space for 2 Dices.

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

$\rightarrow X=0$

$\rightarrow X=1$

$\rightarrow X=2$

$X=1$

Probability Distribution

X	0	1	2
$P(X)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Expection of $X = E(X)$ (mean) $= \sum p_i x_i$

$= p_1 x_1 + p_2 x_2 + p_3 x_3$

$= \cancel{\left(\frac{25}{36} \times 0\right)}^{0} + \left(\frac{10}{36} \cdot 1\right) + \left(\frac{1}{36} \cdot 2\right)$

$= \frac{10+2}{36}$

$= \frac{\cancel{12}^{1}}{\cancel{36}_{3}} = \frac{1}{3}$

Q.12 Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find $E(X)$.

1, 2, 3, 4, 5, 6

2 Numbers (without replacement)

Sample space, $n(S) = 30$

Random Variable $(X) = $ (Larger of the two numbers)

$X = 2, 3, 4, 5, 6$

$X=2$	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,1)	(2,3)	(2,4)	(2,5)	(2,6)
$X=3$	(3,1)	(3,2)	(3,4)	(3,5)	(3,6)
$X=4$	(4,1)	(4,2)	(4,3)	(4,5)	(4,6)
$X=5$	(5,1)	(5,2)	(5,3)	(5,4)	(5,6)
$X=6$	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)

$P(X=2) = \frac{2}{30}$

$P(X=3) = \frac{4}{30}$

$P(X=4) = \frac{6}{30}$

$P(X=5) = \frac{8}{30}$

$P(X=6) = \frac{10}{30}$

Probability Distribution

X	2	3	4	5	6	$\leftarrow x_i$
$P(X)$	$2/30$	$4/30$	$6/30$	$8/30$	$10/30$	$\leftarrow P_i$

$$E(X) = \Sigma(P_i x_i) = \left(2 \cdot \frac{2}{30}\right) + \left(3 \cdot \frac{4}{30}\right) + \left(4 \cdot \frac{6}{30}\right) + \left(5 \cdot \frac{8}{30}\right) + \left(6 \cdot \frac{10}{30}\right)$$

$$= \frac{4 + 12 + 24 + 40 + 60}{30}$$

$$= \frac{140}{30} = \frac{14}{3}$$

Exercise 13.4 — Probability Distribution of Random Variable.

X	x_1	x_2	x_3	$\dots$	x_n
$P(X)$	p_1	p_2	p_3	$\dots$	p_n

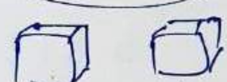
Mean $= E(X) = \sum p_i x_i$

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

$E(X^2) = \sum p_i x_i^2$.

Standard Deviation $(\sigma_x) = SD = \sqrt{var(X)}$

Q.13 Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and Standard deviation of X.



$X =$ Sum of numbers obtained from two Dice.

1, 2, 3, 4, 5, 6 + 1, 2, 3, 4, 5, 6

$n(S) = 36$

(1,1)

$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Probability Distribution.

Sum

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $= E(X) = \sum p_i x_i$

$$= \left(\frac{1}{36}\cdot 2\right) + \left(\frac{2}{36}\cdot 3\right) + \left(\frac{3}{36}\cdot 4\right) + \left(\frac{4}{36}\cdot 5\right) + \left(\frac{5}{36}\cdot 6\right) + \left(\frac{6}{36}\cdot 7\right)$$

$$+ \left(\frac{5}{36}\cdot 8\right) + \left(\frac{5}{36}\cdot 9\right) + \left(\frac{3}{36}\cdot 10\right) + \left(\frac{2}{36}\cdot 11\right) + \left(\frac{1}{36}\cdot 12\right)$$

$$= \frac{1}{36}(252) = 7$$

$$E(X^2) = \Sigma\left(P_i(x_i^2)\right)$$

$$= \left[\frac{1}{36}\cdot(2^2) + \frac{2}{36}\cdot(3^2) + \frac{3}{36}(4^2) + \frac{4}{36}(5^2) + \frac{5}{36}(6^2) + \frac{6}{36}(7^2) + \frac{5}{36}(8^2) + \frac{4}{36}(9^2) + \frac{3}{36}(10^2) + \frac{2}{36}(11^2) + \frac{1}{36}(12^2)\right]$$

$$E(X^2) = \frac{1}{36}[1974] = 54.833$$

$$\text{Variance} = Var(X) = E(X^2) - (E(X))^2$$

$$= 54.833 - (7)^2 = 54.833 - 49$$

$$\boxed{Var(X) = 5.833}$$

Standard Deviation (S.D.) (σ_x) $= \sqrt{Var(X)} = \sqrt{5.833}$

$$= 2.415$$

$$\begin{array}{r|l}
 & 2.41 \\
\hline
2 & \overline{05}.\overline{83}\,\overline{30}\,\overline{00} \\
+2 & -4 \\
\hline
44 & 183 \\
4 & -176 \\
\hline
481 & 730 \\
1 & -481 \\
\hline
\end{array}$$

$$\rightarrow \sqrt{5.833} = \underline{2.41}$$

(14) A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19, and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? ~~ ~~ find mean, variance, SD of X.

X = age of the selected student.

X	14	15	16	17	18	19	20	21	
P(x)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	

Probability Distribution

$$\text{mean} = E(x) = \sum p_i x_i = \left(14 \cdot \frac{2}{15}\right) + 15\left(\frac{1}{15}\right) + \left(16 \cdot \frac{2}{15}\right) + \left(17 \cdot \frac{3}{15}\right) + \left(18 \cdot \frac{1}{15}\right) + \left(19 \cdot \frac{2}{15}\right) + \left(20 \cdot \frac{3}{15}\right) + \left(21 \cdot \frac{1}{15}\right) = \frac{1}{15}(263) = 17.533$$

$$E(x^2) = \sum p_i x_i^2 = \left(14^2 \cdot \frac{2}{15}\right) + \left(15^2 \cdot \frac{1}{15}\right) + \left(16^2 \cdot \frac{2}{15}\right) + \left(17^2 \cdot \frac{3}{15}\right) + \left(18^2 \cdot \frac{1}{15}\right) + \left(19^2 \cdot \frac{2}{15}\right) + \left(20^2 \cdot \frac{3}{15}\right) + \left(21^2 \cdot \frac{1}{15}\right)$$

$$= \frac{1}{15}(4683) = 312.2$$

$$Var(x) = E(x^2) - (E(x))^2 = (312.2) - (17.533)^2$$

$$= 312.2 - 307.417$$

$$= 4.783$$

$$\text{Standard Deviation of } X = SD(x) = \sqrt{4.783} = 2.187$$

2.19

Q.15 In a meeting, 70% of the members favours and 30% oppose a certain proposal. A member is selected at random and we take $X=0$ if he opposed, and $X=1$ if he is in favour. Find $E(X)$ & $Var(X)$.

Ans:

70% → favour ——→ $X=1$ → $P(X=1)=0.7$

30% → oppose ——→ $X=0$ → $P(X=0)=0.3$

Table

X	0	1
$P(X)$	0.3	0.7

$$\text{Mean} = E(X) = \Sigma p_i x_i = (0.3)\times(0) + (0.7)\times(1)$$

$$= 0 + 0.7 = 0.7$$

$$E(X^2) = \Sigma p_i x_i^2 = (0.3).(0^2) + (0.7).(1)^2 = 0.7$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= (0.7) - (0.7)^2$$

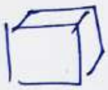
$$= 0.7 - 0.49$$

$$= 0.21$$

$$\begin{array}{r} 0.70 \\ -0.49 \\ \hline 0.21 \end{array}$$

Q.16 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face, is —

(A) 1 (B) 2 (C) 5 (D) $\frac{8}{3}$

 $\Rightarrow \{1, 1, 1, 2, 2, 5\}$

$X =$ number obtained

$X = 1, 2, 5$

X	1	2	5
$P(X)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

(mean)

$$E(X) = \Sigma p_i x_i$$

$$= \left(1 \times \frac{3}{6}\right) + \left(2 \times \frac{2}{6}\right) + \left(5 \times \frac{1}{6}\right)$$

$$= \frac{3 + 4 + 5}{6}$$

$$\Rightarrow \boxed{E(X) = \frac{12}{6} = 2}$$

Q.17 Suppose that two cards are drawn at random from a deck of cards. let X be the number of aces obtained. then the value of $E(X)$ is —

mean

(A) $\frac{37}{221}$ (B) $\frac{5}{13}$ (C) $\frac{1}{13}$ ✓(D) $\frac{2}{13}$

4 Aces
48 Non Ace
52 Cards

$\to$ 2 Cards

P&C

$X =$ No. of Aces

$X = 0, 1, 2$

$P(X=0) = P(\text{No. of Aces} = 0)$

$= P(\text{both non ace})$

$= \frac{{}^{48}C_2}{{}^{52}C_2} = \frac{\left(\frac{48\times 47}{2\times 1}\right)}{\left(\frac{52\times 51}{2\times 1}\right)} = \frac{48\cdot 47}{52\cdot 51} \to P_1$

$P(X=1) = P(\text{No. of aces} = 1)$

$= \frac{{}^{4}C_1 \cdot {}^{48}C_1}{{}^{52}C_2} = \frac{4\times 48}{(52\times 51)} \times 2$

$P(X=2) = P(\text{No. of aces} = 2)$

$= \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{\left(\frac{4(3)}{2\times 1}\right)}{\left(\frac{52\cdot 51}{2\times 1}\right)} = \frac{4\times 3}{52\cdot 51}$

$^nC_r = \frac{n!}{(n-r)!\,r!}$

no. of ways to select r objects out of n-objects

$^nC_2 = \frac{n!}{(n-2)!\,2!}$

$= \frac{n\cdot(n-1)\cdot\cancel{(n-2)!}}{\cancel{(n-2)!}\,2\times 1}$

$= \frac{n(n-1)}{2\times 1}$

$^nC_1 = \frac{n!}{(n-1)!\,1!}$

$= \frac{n\,\cancel{(n-1)!}}{\cancel{(n-1)!}\times 1}$

$= n$

mean $(E(X)) = \cancel{0(P_1)}^{\,0} + \frac{4.48.2}{52\cdot 51} + \frac{4.2.3}{52\cdot 51}$

$= \frac{\cancel{4}^{1}\times 2}{\cancel{52}_{13}\times 51}(48+3) = \frac{2\times \cancel{51}}{13\times \cancel{51}} = \frac{2}{13}$

Bernoulli Trials and Binomial Distribution

$\boxed{\text{Bernoulli Trials}}$ → A random experiment is called bernoulli trials, if they satisfy following condⁿ :→

(i) there should be finite no. of trials

(ii) the trials should be independent

(iii) each trial has exactly two outcomes < Success / Failure

(iv) the probability of success should remain constant.

e.g. ① Tossing a coin 50 times ✓ Bernoulli Trials

② Tossing a die 100 times ✓

✗ ③ Drawing 5 cards from a deck of 52 cards without replacement. → probability → change

Not a Bernoulli trial

✓ ④ Drawing 5 cards from a deck of 52 cards with replacement.

✗ ⑤ Tossing a coin for infinite times.

Binomial Distribution

[Probability Distribution for Bernoulli Trials]

Probability of x successes in n-trials

$$= P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

$${}^{n}C_{x} = \frac{n!}{(n-x)!\,x!}$$

$p+q=1$

where

- n = no. of bernoulli trials
- x = no. of Success
- p = Probability of Success
- q = prop. of failure $= 1-p$

Binomial Distribution ⇓ $B(n,p)$

e.g. If a fair coin is tossed 10 times, find the probability of (i) exactly six heads

(ii) atleast six heads

(iii) atmost six heads.

Ans. $n=10$, Success = getting Head

p = Prob. of Success $\Rightarrow \boxed{p = \frac{1}{2}}$

q = Prob. of failure $\Rightarrow \boxed{q = \frac{1}{2}}$

$q = 1-p$

Prob. of x successes in 10 trials.

$$P(x) = {}^{10}C_{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{10-x} = {}^{10}C_{x} \left(\frac{1}{2}\right)^{\cancel{x}+10-\cancel{x}}$$

$$\boxed{P(x) = {}^{10}C_{x} \cdot \left(\frac{1}{2}\right)^{10}}$$

(i) Probability of exactly 6 heads

6 → Success.

$P(x) = {}^{10}C_x \cdot \left(\frac{1}{2}\right)^{10}$ $x = 6$

${}^nC_r = \frac{n!}{(n-r)!\,r!}$

$P(6) = {}^{10}C_6 \cdot \left(\frac{1}{2}\right)^{10}$

${}^{10}C_6 = \frac{10!}{4!\,6!}$

$= \cancel{(210)}^{105} \times \frac{1}{\cancel{(2^{10})}_{2^9}}$

$= \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \times \cancel{6!}}$

$= \frac{105}{2^9} = \frac{105}{512}$

$= 210$

$\frac{\overset{105}{\cancel{210}}}{\cancel{2}}$

(ii) atleast six heads → Success.

$\boxed{P(x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10}}$

6, 7, 8, 9, 10

$P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$

$= {}^{10}C_6 \cdot \left(\frac{1}{2}\right)^{10} + {}^{10}C_7 \left(\frac{1}{2}\right)^{10} + {}^{10}C_8 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10}$
$+ {}^{10}C_{10} \cdot \left(\frac{1}{2}\right)^{10}$

$= \left(\frac{1}{2}\right)^{10} \cdot \left\{ {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right\} = \frac{386}{2^{10}}$

${}^{10}C_6 \to 210$

${}^{10}C_7 \to \frac{10!}{3!\,7!} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot \cancel{7!}}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{7!}} \to 120.$

${}^{10}C_8 \to \frac{10!}{2!\,8!} = \frac{\overset{5}{\cancel{10}} \cdot 9 \cdot \cancel{8!}}{\cancel{2} \times 1 \times \cancel{8!}} \to 45$

${}^{10}C_9 \to \frac{10!}{9!\,1!}$, $1! \to 1$, $\frac{10 \times \cancel{9!}}{\cancel{9!}} \to 10$

${}^{10}C_{10} \to \frac{\cancel{10!}}{0!\,\cancel{10!}} = 1$, $0! \to 1$

(iii) at most 6 heads

$P(x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

0 1 2 3 4 5 6

$P(x \le 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$

$= {}^{10}C_0 \cdot \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + \dots + {}^{10}C_6 \left(\frac{1}{2}\right)^{10}$

$= \left(\frac{1}{2}\right)^{10} \left\{ {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 \right\}$

${}^{10}C_0 \to 1$, ${}^{10}C_1 \to {}^{10}C_9 \to 10$, ${}^{10}C_2 \to {}^{10}C_8 \to 45$, ${}^{10}C_3 \to {}^{10}C_7 \to 120$, ${}^{10}C_4 \to 210$, ${}^{10}C_5 \to 252$, ${}^{10}C_6 \to \cancel{2110}$ 210

${}^nC_r = {}^nC_{n-r}$

${}^{10}C_6 = {}^{10}C_{10-6} = {}^{10}C_4$

${}^nC_0 = 1 = {}^nC_n$

$= \left(\frac{1}{2}\right)^{10} \times (484)$

$= \frac{484}{1024}$

e.g. 10 eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the prob. that there is atleast one defective egg.

$n = 10$ Success = getting defective egg. X = no. of Success

$\boxed{P = 10\% = \frac{10}{100} = 0.1 = \frac{1}{10}}$; $\boxed{q = 1 - P = 1 - \frac{1}{10} = \frac{9}{10}}$

ATQ. $P(x \ge 1) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10)$

$P(x) = {}^nC_x . P^x . q^{n-x}$

$= 1 - P(0)$

$= 1 - \cancel{{}^{10}C_0 \cdot \left(\frac{1}{10}\right)^0} \cdot \left(\frac{9}{10}\right)^{10-0}$

${}^{10}C_0 = 1$

$\left(\frac{1}{10}\right)^0 = 1$

$= 1 - \left(\frac{9}{10}\right)^{10}$

Exercise 13.5 (Bernoulli Trials & Binomial Distribution)

Probability of x-successes in n-trials

$$= P(x) = {}^nC_x \cdot p^x \cdot q^{n-x}$$

where

n = no. of trials

p = prob. of success

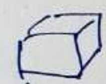
q = prob. of failure $= 1-p$

x = no. of successes

$$\boxed{{}^nC_x = \frac{n!}{(n-x)!\,x!}}$$

$$p+q=1$$

Q.1 A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of (i) 5 success? (ii) atleast 5 success? (iii) at most 5 successes?



$\{1, 2, 3, 4, 5, 6\}$

$n = 6$; success = getting an odd no.

p = prob. of success $= \frac{3}{6} = \frac{1}{2}$

$q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

Prob. of x successes in 6-trials $= {}^nC_x \cdot p^x \cdot q^{n-x}$

$$P(x) = {}^6C_x \cdot \left(\tfrac{1}{2}\right)^x \cdot \left(\tfrac{1}{2}\right)^{6-x} = {}^6C_x \cdot \left(\tfrac{1}{2}\right)^{x+6-x}$$

$$\boxed{P(x) = {}^6C_x \cdot \left(\tfrac{1}{2}\right)^6}$$

(i) 5 successes.

$(x=5)$

$$P(5) = {}^6C_5 \left(\tfrac{1}{2}\right)^6$$

$$= \cancel{6}^{3} \times \frac{1}{\cancel{64}_{32}} = \frac{3}{32}$$

$${}^6C_5 = \frac{6!}{1!\,.5!} = \frac{6 \times \cancel{5!}}{\cancel{5!}} = 6$$

(ii) at least 5 successes $\quad$ $\boxed{x \le n}$ $\quad$ $\boxed{{}^nC_n = 1 = {}^nC_0}$

$(x \geq 5)$ $\quad$ no. of success $\quad$ no. of trials

$x = 5,\ x = 6$

$P(x) = {}^6C_x \left(\frac{1}{2}\right)^6$

$$P(x \geq 5) = P(5) + P(6)$$

$$= {}^6C_5\left(\frac{1}{2}\right)^6 + {}^6C_6\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^6 \cdot \left\{ \underbrace{{}^6C_5}_{6} + \underbrace{{}^6C_6}_{1} \right\}$$

$$= \left(\frac{1}{2}\right)^6 \times 7 = \frac{7}{64}$$

(iii) atmost 5 successes ?

$x \le 5$ $\quad$ x $\quad$ $P(x) = {}^6C_{x} \left(\frac{1}{2}\right)^6$

$x = 0, 1, 2, 3, 4, 5$

$$P(x \le 5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= 1 - P(6)$$

$$= 1 - {}^6C_6\left(\frac{1}{2}\right)^6$$

$${}^6C_6 = 1 = {}^nC_n$$

$$= 1 - 1 \cdot \left(\frac{1}{64}\right) = 1 - \frac{1}{64} = \frac{63}{64}$$

$\boxed{Q.2}$ A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two success.

$n = 4$

$(1,1), (2,2), \ldots, (6,6)$

$p = \frac{6}{36} = \frac{1}{6}$

Success = getting doublet

$= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$q = 1 - p = 1 - \frac{1}{6}$

$= \frac{5}{6}$

$x = 2$ = no. of success

Probability of 2 successes in 4 throws

$= P(2) = {}^nC_x \, p^x \cdot q^{n-x} = {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{4-2}$

$= \frac{4!}{2!\,2!} \cdot \frac{1}{6^2} \cdot \frac{5^2}{6^2}$

$= \frac{\cancel{4}^{2} \times 3 \times \cancel{2 \times 1}}{\cancel{2} \times 1 \times \cancel{2 \times 1}} = \cancel{6} \times \frac{1}{\cancel{6^2}_{6}} \times \frac{25}{6^2} = \frac{25}{216}$

Q.3 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

$n =$ no. of trials $= 10$

(Success = getting defective item)

$p =$ prob. of success $= \frac{\cancel{5}^{1}}{\cancel{100}_{20}} = \frac{1}{20}$

$x =$ no. of success

$0 \le x \le 1$

$x \neq 1$

$q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

$x = 0, 1$

$P(x) = {}^nC_x \, p^x q^{n-x}$

$P(x \le 1) = P(x=0) + P(x=1)$

${}^{10}C_x \cdot \left(\frac{1}{20}\right)^x \cdot \left(\frac{19}{20}\right)^{10-x}$

$= \left[{}^{10}C_0 \cdot \left(\frac{1}{20}\right)^0 \times \left(\frac{19}{20}\right)^{10-0}\right] +$

$\left[{}^{10}C_1 \cdot \left(\frac{1}{20}\right)^1 \cdot \left(\frac{19}{20}\right)^9\right]$

${}^{10}C_1 = 10$

${}^nC_1 = n$

$= 1 \times 1 \times \left(\frac{19}{20}\right)^{10} + 1\cancel{0} \times \frac{1}{2\cancel{0}} \times \left(\frac{19}{20}\right)^9$

$= \left(\frac{19}{20}\right)^9 \left\{\frac{19}{20} + \frac{10}{20}\right\} = \left(\frac{19}{20}\right)^9 \cdot \left(\frac{29}{20}\right)$

Q.4 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(i) all the 5 cards are spade (ii) only 3 cards are spade. (iii) none is a spade.

n = no. of trials = 5.

$\boxed{\text{Success } (x) = \text{getting spade}}$

p = Prob. of success $= \frac{\cancel{13}^{1}}{\cancel{52}_{4}} = \frac{1}{4}$

$q = 1 - p = 1 - \frac{1}{4} = 3/4$

$P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$

$P(x) = {}^{5}C_{x} \cdot \left(\frac{1}{4}\right)^{x} \cdot \left(\frac{3}{4}\right)^{5-x}$

(i) all the 5 cards are spade.

$(x = 5)$ $P(5) = {}^{5}C_{5} \cdot \left(\frac{1}{4}\right)^{5} \cdot \left(\frac{3}{4}\right)^{5-5} = 1 \times \frac{1}{4^5} \times 1$

$\boxed{P(5) = \frac{1}{1024}}$

(ii) ~~of~~ only 3 cards are spade.

$(x = 3)$ $P(x=3) = {}^{5}C_{3} \left(\frac{1}{4}\right)^{3} \left(\frac{3}{4}\right)^{5-3}$

$= \frac{5!}{2!\,3!} \cdot \frac{1}{(4^3)} \cdot \frac{3^2}{4^2} = \frac{5 \times \cancel{4} \times \cancel{3!}}{\cancel{2} \times 1 \times \cancel{3!}} \cdot \frac{9}{\cancel{4^5}\,4^4}$

$= \frac{45}{2 \times 256} = \frac{45}{512}$

${}^{n}C_{0} = 1 = {}^{n}C_{n}$

(iii) none is a spade. $(x = 0)$ $P(x=0) = {}^{5}C_{0} \left(\frac{1}{4}\right)^{0} \cdot \left(\frac{3}{4}\right)^{5-0}$

$= 1 \times 1 \times \frac{243}{1024} = \frac{243}{1024}$

Q.5 The probability that a bulb produced by a factory will fuse after ~~180~~ days of use is 0.05. Find the probability that out of $\boxed{\text{5 such bulbs}}$ (i) none (ii) not more than one (iii) more than one (iv) at least one, will fuse after ~~150~~ days of use.

n = no. of trial = 5

Success = a bulb will fuse after 150 days of use.

p = prop. of success = 0.05

$q = 1 - p = 1 - 0.05 = 0.95$

x = no. of success

$$\boxed{P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x} = {}^{5}C_{x} \cdot (0.05)^{x} \cdot (0.95)^{5-x}}$$

(i) none will fuse,

$(x=0)$ $P(0) = {}^{5}C_{0} \cdot (0.05)^{0} \cdot (0.95)^{5-0} = (0.95)^{5}$

1 1

(ii) not more than one $P(x \le 1) = P(0) + P(1)$

$0 \le x \le 1$ $x \ne 1$ $x = 0, 1$

$= (0.95)^{5} + {}^{5}C_{1} (0.05)^{1} \cdot (0.95)^{4}$

$= (0.95)^{4} \cdot \{0.95 + 5 \cdot (0.05)\}$

${}^{n}C_{1} = n$

$= (0.95)^{4} \cdot (0.95 + 0.25)$

$= (0.95)^{4} \times (1.2)$

$$\begin{array}{r} .95 \\ .25 \\ \hline 1.20 \end{array}$$

(iii) more than one will fuse after 15

$x > 1$

$x = 2, 3, 4, 5$

5

0, 1, 2, 3, 4, 5

$P(x > 1) = 1 - P(x \le 1)$

$$\boxed{P(x) = {}^{5}C_{x} (0.05)^{x} \cdot (0.95)^{5-x}}$$

$= 1 - [P(0) + P(1)]$

$= 1 - (0.95)^4 \times 1.2$ (by previous part)

(iv) at least one will fuse

$x \geq 1$

0, $\boxed{1, 2, 3, 4, 5}$?

$P(x \geq 1) = 1 - P(0)$

$= 1 - (0.95)^5$ by Part (i)

Exercise 13.5 — Bernoulli Trials & Binomial Distribution

$B(n, p)$

Probability of getting x-successes in n-trials

$$= P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

where

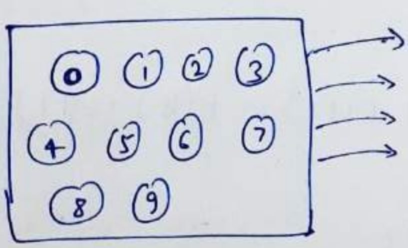
n = no. of trials

p = prob. of success

q = prob. of failure $= 1-p$

x = no. of success

Q.6 A bag consists of 10 Balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?



4 balls (with replacement)

n = no. of trials $= 4$

Success = (getting the ball marked with digit '0')

p = Prob. of success $= \frac{1}{10}$

q = prob. of failure $= 1-p = 1-\frac{1}{10} = \frac{9}{10}$

$${}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

$$P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-1} = {}^{4}C_{x}\left(\frac{1}{10}\right)^{x} \cdot \left(\frac{9}{10}\right)^{4-x}$$

P(no ball is marked with digit '0') $= P(x=0)$

$${}^{n}C_{0} = 1$$
$${}^{n}C_{n} = 1$$

$$= {}^{4}C_{0} \cdot \left(\frac{1}{10}\right)^{0} \cdot \left(\frac{9}{10}\right)^{4-0}$$

$$= 1 \times 1 \times \left(\frac{9}{10}\right)^{4} = \left(\frac{9}{10}\right)^{4}$$

Q.7 In an examination, 20 Questions of True-False type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability the answers at least 12 questions correctly.

Ans.

$H \rightarrow$ True $\nearrow \checkmark$, $\searrow \times$

$T \rightarrow$ False $\nearrow \checkmark$, $\searrow \times$

✓ n = no. of trials = 20

✓ Success = answering correctly

✓ $p = \frac{1}{2}$

✓ $q = \frac{1}{2} = (1-p)$

$\boxed{\text{no. of success} = x = (\text{atleast } 12) \quad x \geq 12}$

✓ $P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x} = {}^{20}C_{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{20-x} = {}^{20}C_{x} \cdot \left(\frac{1}{2}\right)^{\cancel{x}+20-\cancel{x}}$

$$\boxed{P(x) = {}^{20}C_{x} \cdot \left(\frac{1}{2}\right)^{20}}$$

$P(x \geq 12) = P(12) + P(13) + P(14) + P(15) + P(16) + P(17) + P(18) + P(19) + P(20)$

$= {}^{20}C_{12} \cdot \left(\frac{1}{2}\right)^{20} + {}^{20}C_{13}\left(\frac{1}{2}\right)^{20} + \cdots + {}^{20}C_{20} \cdot \left(\frac{1}{2}\right)^{20}$

$= \left(\frac{1}{2}\right)^{20} \cdot \left\{ {}^{20}C_{12} + {}^{20}C_{13} + {}^{20}C_{14} + {}^{20}C_{15} + {}^{20}C_{16} + {}^{20}C_{17} + {}^{20}C_{18} + {}^{20}C_{19} + {}^{20}C_{20} \right\}$

Q.8 Suppose X has a binomial distribution $B(6, \frac{1}{2})$. Show that $X = 3$ is the most likely outcome.

(Hint: $P(X=3)$ is the maximum among all $P(x_i)$, $i = 0,1,2,3,4,5,6$)

Ans. $B(n, p) \equiv B(6, \frac{1}{2})$

X = no. of success.

$x = 0, 1, 2, 3, 4, 5, 6$

$n = 6$, $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

no. of trials

To prove: $P(X=3)$ is maximum.

$$\boxed{P(x) = {}^{n}C_{x} (p)^{x} (q)^{n-x}}$$

$$\Rightarrow P(x) = {}^{6}C_{x} \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{6-x} = {}^{6}C_{x} \left(\frac{1}{2}\right)^{\cancel{x}+6-\cancel{x}}$$

$$\Rightarrow \boxed{P(x) = {}^{6}C_{x} \left(\frac{1}{2}\right)^{6} = \frac{1}{64} \cdot {}^{6}C_{x}}$$

$${}^{n}C_{r} = \frac{n!}{(n-r)!\, r!}$$

$$\boxed{{}^{n}C_{0} = 1 = {}^{n}C_{n}}$$

$$\boxed{{}^{n}C_{1} = n = {}^{n}C_{n-1}}$$

$${}^{n}C_{r} = {}^{n}C_{n-r}$$

$${}^{6}C_{2} = {}^{6}C_{6-2}$$

$P(x=0) = \frac{1}{64} \cdot {}^{6}C_{0} = \frac{1}{64} \times 1$

$P(x=1) = \frac{1}{64} \cdot {}^{6}C_{1} = \frac{1}{64} \times 6$

$P(x=2) = \frac{1}{64} \cdot {}^{6}C_{2} = \frac{1}{64} \times 15$

$\boxed{P(x=3) = \frac{1}{64} \cdot {}^{6}C_{3} = \frac{1}{64} \times 20}$ → max. prob.

$P(x=4) = \frac{1}{64} \cdot {}^{6}C_{4} = \frac{1}{64} \times 15$

$P(x=5) = \frac{1}{64} \cdot {}^{6}C_{5} = \frac{1}{64} \times 6$

$P(x=6) = \frac{1}{64} \cdot {}^{6}C_{6} = \frac{1}{64} \cdot 1$

$${}^{6}C_{2} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot 2 \times 1} = 15$$

$${}^{6}C_{3} = \frac{6!}{3!\,3!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\, 3 \cdot 2 \cdot 1} = \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{3 \cdot 2 \cdot 1}} = 20$$

Q.9 On a multiple choice examination with three possible answers for each of the 5 questions, what is the probability that a candidate ~~was~~ would get four or more correct answers just by guessing?

Ans. 5 questions $\boxed{n=5}$ — (Success = answering correct by guessing)

Q. A, B, C

$p = \frac{1}{3}$

$q = 1-p = 1-\frac{1}{3} = \frac{2}{3}$

$P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x} = {}^{5}C_{x}\left(\frac{1}{3}\right)^{x} \cdot \left(\frac{2}{3}\right)^{5-x}$

$P(x \geq 4) = P(x=4) + P(x=5)$

$= {}^{5}C_{4} \cdot \left(\frac{1}{3}\right)^{4}\left(\frac{2}{3}\right)^{5-4} + {}^{5}C_{5} \cdot \left(\frac{1}{3}\right)^{5}\left(\frac{2}{3}\right)^{5-5}$

$= \left[5 \times \frac{1}{3^4} \cdot \frac{2}{3}\right] + \left[\frac{1}{3^5} \cdot 1\right]$

$= \frac{10+1}{3^5}$

$= \frac{11}{243}$

${}^{5}C_{4} = \frac{5!}{1!\,4!} = \frac{5 \times \cancel{4!}}{1 \times \cancel{4!}}$

${}^{5}C_{5} = 1$

Q.10 A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize (a) at least once (b) exactly once (c) at least twice.

$n = 50$; Success = he will win a prize.

$p = \frac{1}{100}$; $q = 1-p = 1-\frac{1}{100} = \frac{99}{100}$

$P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$

$P(x) = {}^{50}C_{x} \cdot \left(\frac{1}{100}\right)^{x} \cdot \left(\frac{99}{100}\right)^{50-x}$

(a) at least once.

$x \geq 1$

x = no. of success

0, 1, 2, ..., 50

$P(x \geq 1) = P(1) + P(2) + \cdots + P(50)$

$= 1 - P(0) = 1 - {}^{50}C_0 \cdot \left(\frac{1}{100}\right)^0 \cdot \left(\frac{99}{100}\right)^{50-0}$

1 1

$= 1 - \left(\frac{99}{100}\right)^{50}$

${}^{n}C_0 = 1$

(b) exactly once $P(x=1) = {}^{50}C_1 \cdot \left(\frac{1}{100}\right)^1 \cdot \left(\frac{99}{100}\right)^{50-1}$

$\boxed{x=1}$

${}^{n}C_1 = n$

$= \cancel{50}^{1} \cdot \frac{1}{\cancel{100}_{2}} \cdot \left(\frac{99}{100}\right)^{49} = \frac{1}{2} \cdot \left(\frac{99}{100}\right)^{49}$

(c) at least twice x = no. of success (win) = 0, 1, 2, ..., 50

$\boxed{x \geq 2}$

$P(x \geq 2) = P(2) + P(3) + P(4) + \cdots + P(50)$

$= 1 - [P(0) + P(1)]$

$= 1 - \left[\left(\frac{99}{100}\right)^{50} + \frac{1}{2}\left(\frac{99}{100}\right)^{49}\right]$ of {by previous Parts}

$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left\{\frac{99}{100} + \frac{1}{2}\right\}$

$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left\{\frac{99+50}{100}\right\} = 1 - \left(\frac{99}{100}\right)^{49} \cdot \left(\frac{149}{100}\right)$

Exercise 13.5 Binomial Distribution

Probability of getting x-successes in n-trials

$$= P(x) = {}^nC_x \cdot p^x \cdot q^{n-x}$$

where

n = no. of trials

p = prob. of success

q = prob. of failure = $1-p$

x = no. of success

$p+q=1$

Q.11 Find the probability of getting '5' exactly twice in 7 throws of a die.

$\rightarrow \{1, 2, 3, 4, 5, 6\}$

Ans. $\boxed{n=7}$, Success = getting '5' on a Die.

$\boxed{p = \text{prob. of success} = \frac{1}{6}}$; $\boxed{q = 1-p = 1-\frac{1}{6} = \frac{5}{6}}$

x = no. of success = 2 (exactly twice)

$$P(x) = {}^nC_x\, p^x \cdot q^{n-x}$$

$$P(x=2) = {}^7C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{7-2}$$

$$= {}^7\cancel{21} \cdot \frac{1}{\cancel{36}_{12}} \cdot \left(\frac{5}{6}\right)^5$$

$$= \frac{7}{12} \cdot \left(\frac{5}{6}\right)^5$$

$${}^7C_2 = \frac{7!}{5!\,2!} = \frac{7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{5!}\, \cancel{2} \times 1}$$

$$= 7 \times 3 = 21$$

Q.12 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

(x) Success

(n)

$\square = \{1, 2, 3, 4, 5, 6\}$

$n = 6$, Success = getting '6' on a die.

$p = \frac{1}{6}$; $q = 1 - \frac{1}{6} = \frac{5}{6}$; x = no. of success $\rightarrow$ 0, 1, 2, 3, 4, 5, 6

x = at most 2

$\boxed{x \le 2}$

$x = 0, 1, 2$

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$\boxed{P(x) = {}^{6}C_{x} \cdot \left(\frac{1}{6}\right)^{x} \cdot \left(\frac{5}{6}\right)^{6-x}}$$

$$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= {}^{6}C_{0} \cdot \left(\frac{1}{6}\right)^{0} \cdot \left(\frac{5}{6}\right)^{6-0} + {}^{6}C_{1} \cdot \left(\frac{1}{6}\right)^{1} \cdot \left(\frac{5}{6}\right)^{5} + {}^{6}C_{2} \left(\frac{1}{6}\right)^{2} \left(\frac{5}{6}\right)^{4}$$

1   6   15

$$= \left(\frac{5}{6}\right)^{4} \cdot \left\{\left(\frac{5}{6}\right)^{2} + 6 \times \frac{1}{6} \times \frac{5}{6} + 15 \times \frac{1}{36}\right\}$$

$$= \left(\frac{5}{6}\right)^{4} \cdot \left\{\frac{25}{36} + \frac{30}{36} + \frac{15}{36}\right\} = \left(\frac{5}{6}\right)^{4} \cdot \left(\frac{\cancel{70}^{35}}{\cancel{36}_{18}}\right)$$

$$= \frac{35}{18} \cdot \left(\frac{5}{6}\right)^{4}$$

$${}^{6}C_{2} = \frac{6!}{4!\,2!} = \frac{\cancel{6}^{3} \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot \cancel{2} \times 1}$$

$$= 15$$

Q.13 It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Ans. $n = 12$, Success = getting a defective article.

$\boxed{p = \frac{10}{100} = \frac{1}{10}}$

x = no. of success = 9

$\boxed{q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}}$

$P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$

$P(x=9) = {}^{12}C_{9} \cdot \left(\frac{1}{10}\right)^{9} \cdot \left(\frac{9}{10}\right)^{3}$

$= 220 \cdot \frac{1}{(10^9)} \cdot \frac{9^3}{10^3}$

$= 220 \cdot \frac{9^3}{10^{12}} = 22 \cdot \frac{9^3}{10^{11}}$

${}^{12}C_{9} = \frac{12!}{3!\,9!} = \frac{\cancel{12}^{2} \times 11 \cdot 10 \cdot \cancel{9!}}{\cancel{3 \times 2 \times 1} \cdot \cancel{9!}} = 220$

Q.14 In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective, is – (A) 10^{-1} (B) $\left(\frac{1}{2}\right)^5$ (C) $\left(\frac{9}{10}\right)^5$ (D) $\frac{9}{10}$

$n = 5$; Success = defective bulb; x = no. of success $= 0$

$p = \frac{10}{100} = \frac{1}{10}$

$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

$P(x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$

$P(0) = {}^{5}C_{0} \cdot \left(\frac{1}{10}\right)^{0} \cdot \left(\frac{9}{10}\right)^{5}$

1 1

$P(0) = \left(\frac{9}{10}\right)^{5}$

(15) The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of 5 students, four are swimmer is —

$P(\text{Not a swimmer}) = \frac{1}{5}$

(A) $^5C_4 \cdot \left(\frac{4}{5}\right)^4 \cdot \frac{1}{5}$ (B) $\left(\frac{4}{5}\right)^4 \cdot \frac{1}{5}$

$P(\text{Swimmer}) = 1 - \frac{1}{5} = \frac{4}{5}$

(C) $^5C_1 \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^4$ (D) None of these.

✓ $n = 5$, Success = swimmer, x = no. of Success

✓ $p = 4/5$ ✓ $x = 4$

✓ $q = 1/5$

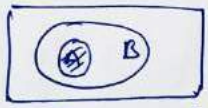
$$P(x) = {^nC_x} \cdot p^x \cdot q^{n-x}$$

$$P(4) = {^5C_4} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{1}{5}\right)^{5-4}$$

$$\boxed{P(4) = {^5C_4} \cdot \left(\frac{4}{5}\right)^4 \cdot \left(\frac{1}{5}\right)}$$

Miscellaneous Exercise on Chapter 13

Q.1 A and B are two events such that $P(A) \neq 0$. Find $P(B|A)$ if (i) A is a subset of B (ii) $A \cap B = \phi$

Ans. (i) $\boxed{A \subset B}$  $A \cap B = A$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

(ii) $A \cap B = \phi$; $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(\phi)}{P(A)} = \frac{0}{P(A)} = 0$$

Q.2 A couple has two children,

(i) Find the probability that (both children are males) $\rightarrow E$, if it is known that (at least one of the children is male.) $\rightarrow H$

Sample space $= S = \{MM, FM, MF, FF\}$

$E = \{MM\}$; $H = \{MM, FM, MF\}$; $E \cap H = \{MM\}$

Question $$P(E|H) = \frac{P(E \cap H)}{P(H)} = \frac{(1/4)}{(3/4)} = \frac{1}{3}$$

(ii) Find the probability that (both children are females,) $\rightarrow A$ if it is known that ~~at least~~ (the elder child is female) $\rightarrow B$

$S = \{MM, FM, MF, FF\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$A = \{FF\}$

$B = \{FM, FF\} \rightsquigarrow P(B) = \frac{2}{4} = \frac{1}{2}$

$A \cap B = \{FF\} \rightsquigarrow P(A \cap B) = \frac{1}{4}$

$$= \frac{(1/4)}{(2/4)} = \frac{1}{2}$$

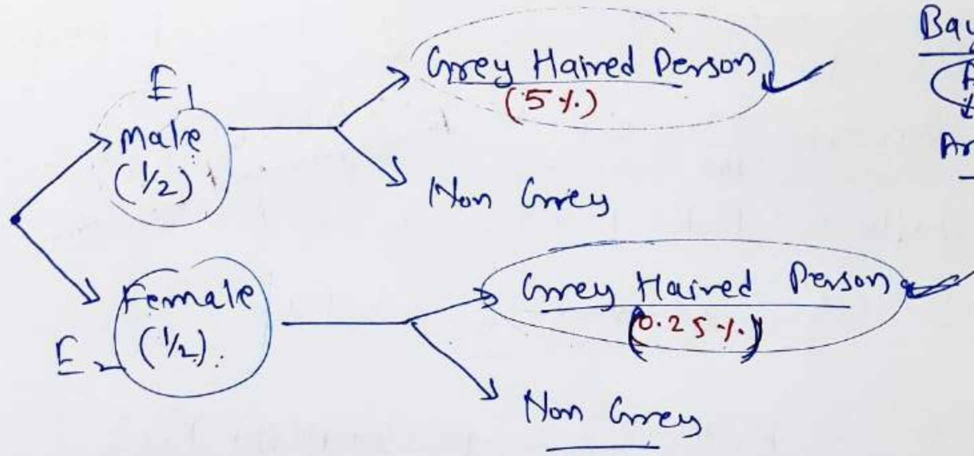
Q3

Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is ~~sel~~ selected at random. What is the probability of this person being male? (Assume that there are equal number of males & females).

Bayes' Theorem

AEPB

Arrow Diagram



E_1: male

E_2: female

A: Grey Haired Person

$P(E_1) = \frac{1}{2}$ $\quad P(A|E_1) = 5\% = \frac{5}{100} = 0.05$

$P(E_2) = \frac{1}{2}$ $\quad P(A|E_2) = 0.25\% = \frac{0.25}{100} = 0.0025$

P(Selected Person is male given that Grey Haired person)

$$= P(E_1|A) = \frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2)}$$

(Bayes' Theorem)

$$= \frac{\cancel{\tfrac{1}{2}}(0.05)}{\cancel{\tfrac{1}{2}}(0.05) + \cancel{\tfrac{1}{2}}(0.0025)} = \frac{0.0500}{0.0525} = \frac{\overset{20}{\cancel{500}}}{\underset{21}{\cancel{525}}}$$

$$= \frac{20}{21}$$

Q. 4 Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random ~~pe~~ sample of 10 people are right-handed?

$$\boxed{P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}}$$

Success = getting right-handed person

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

n = no. of trials = 10

p = prob. of success = $\frac{90}{100} = 0.9$

q = prob. of failure = $1 - p = 1 - 0.9 = 0.1$

x = no. of success = at most 6 $\Rightarrow \boxed{x \le 6}$

$$P(\text{at most } 6) = P(x \le 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$$

या

$$= 1 - \left[P(7) + P(8) + P(9) + P(10)\right]$$

$$= 1 - \sum_{x=7}^{10} P(x) = 1 - \sum_{x=7}^{10} {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$$

$$= 1 - \sum_{x=7}^{10} \left({}^{10}C_{x} \cdot (0.9)^{x} \cdot (0.1)^{10-x} \right).$$

$n = 10$, $p = 0.9$, $q = 0.1$

$$= 1 - \sum_{r=7}^{10} \left({}^{10}C_{r} \cdot (0.9)^{r} \cdot (0.1)^{10-r} \right)$$

Q.5 An urn contains 25 balls of which 10 balls bear a mark 'X' & the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(i) all will bear 'X' mark
(ii) not more than '2' will bear 'Y' mark.
(iii) at least one ball will bear 'Y' mark
(iv) the number of balls with 'X' and 'Y' will be equal.

'X' 10	
'Y' 15	
25 total	

X/Y with replacement

6 balls

$\boxed{n=6}$

Success = getting X mark

$P(x) = {}^nC_x\, p^x \cdot q^{n-x}$

$p = \frac{10}{25} = \frac{2}{5}$

$q = 1-p = 1-\frac{2}{5} = 3/5$

$\boxed{P(x) = {}^6C_x \cdot \left(\frac{2}{5}\right)^x \cdot \left(\frac{3}{5}\right)^{6-x}}$

${}^nC_x = \frac{n!}{(n-x)!\,x!}$

(i) all will bear 'X' mark

x = no. of successes $= 6$

$P(6) = {}^6C_6 \left(\frac{2}{5}\right)^6 \cdot \left(\frac{3}{5}\right)^{6-6} = 1 \cdot \left(\frac{2}{5}\right)^6 \cdot \left(\frac{3}{5}\right)^0 = \left(\frac{2}{5}\right)^6$

$P(X=6)$

(ii) not more than '2' will bear 'y' mark

$y \le 2 \rightarrow y = 0, 1, 2$

$x = 6, 5, 4.$

$P(y \le 2) = P(x \ge 4) = P(x=4) + P(x=5) + P(x=6)$

$$\boxed{P(x) = {}^6C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{6-x}}$$

$$= P(4) + P(5) + P(6)$$

$$= \underbrace{{}^6C_4}_{15} \left(\frac{2}{5}\right)^4 \cdot \left(\frac{3}{5}\right)^2 + \underbrace{{}^6C_5}_{6} \left(\frac{2}{5}\right)^5 \cdot \left(\frac{3}{5}\right)^1 + \underbrace{{}^6C_6}_{1} \left(\frac{2}{5}\right)^6 \left(\frac{3}{5}\right)^0$$

$$\cancel{= 15 \cdot \frac{2}{5}}$$

$$= \left(\frac{2}{5}\right)^4 \left\{ 15 \cdot \frac{9}{25} + 6 \cdot \frac{2}{5} \cdot \frac{3}{5} + \frac{4}{25} \right\}$$

$$= \left(\frac{2}{5}\right)^4 \cdot \left\{ \frac{135 + 36 + 4}{25} \right\}$$

$$= \left(\frac{2}{5}\right)^4 \left\{ \frac{\cancel{175}^{\,7}}{\cancel{25}} \right\} = 7 \left(\frac{2}{5}\right)^4$$

$${}^6C_4 = \frac{6!}{2!\,4!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{2 \times 1 \cdot \cancel{4!}} = 15$$

$${}^6C_5 = \frac{6!}{1!\,5!} = \frac{6 \times \cancel{5!}}{1 \times \cancel{5!}}$$

(iii) at least one ball will bear 'y' mark.

$y \geq 1$ $\quad y = 1, 2, 3, 4, 5, 6$ $\quad$ total $= 6$

$x = 5, 4, 3, 2, 1, 0 \rightarrow$ formula

$$P(y \geq 1) = P(x=5) + P(x=4) + P(x=3) + P(x=2) + P(x=1) + P(x=0)$$

$0, 1, 2, 3, 4, 5, 6$

$$= 1 - P(x=6)$$

$$= 1 - \left(\frac{2}{5}\right)^6 \quad \text{(by Part (i))}$$

(iv) the number of balls with 'x' mark & 'y' mark will be equal.

$x = y$ $\qquad$ $x + y = 6$

$P(x=3) = {}^6C_x \left(\frac{2}{5}\right)^x \cdot \left(\frac{3}{5}\right)^{6-x}$

$= {}^6C_3 \left(\frac{2}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^{6-3}$

$= 20 \times \left(\frac{2}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^3$

$= \cancel{20}^{4} \times \frac{(6)^3}{\cancel{5^6}\ 5^5}$

$= \frac{4 \times 216}{3125} = \frac{864}{3125}$

$x + x = 6$

$2x = 6$

$\boxed{x = 3}$

${}^6C_3 = \frac{6!}{3!\,3!}$

$= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1} \cdot 3 \cdot 2 \cdot 1}$

$= 20$

$5^4 = 625$

$5^5 = 625 \times 5 = 3125$

Miscellaneous Exercise on Chapter 13

Q.6 In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. What is the probability that he will knock down fewer than 2 hurdles?

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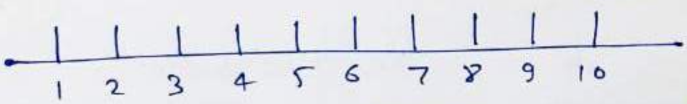


Bernoulli Trials
(Binomial Distribution)

$n = 10$

$$\boxed{P(x) = {}^{n}C_x \, p^x \, q^{n-x}}$$

$P(\text{Clear}) = \frac{5}{6}$

$P(\text{Knock down}) = \frac{1}{6}$

Success = To knock down a Hurdle.

$p = \frac{1}{6}$; $q = \frac{5}{6}$

$$\boxed{P(x) = {}^{10}C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}}$$

Prob. of 'x' successes

x = no. of successes.

x = fewer than 2

$x < 2$

$\boxed{x = 0, 1}$

$$P(x<2) = P(x=0) + P(x=1)$$

$$= {}^{10}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9$$

$$= 1 \cdot 1 \left(\frac{5}{6}\right)^{10} + \frac{10}{6} \cdot \left(\frac{5}{6}\right)^9$$

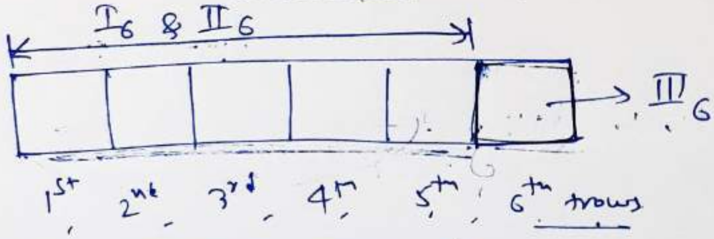
$$= \left(\frac{5}{6}\right)^9 \cdot \left\{\frac{5}{6} + \frac{10}{6}\right\} = \left(\frac{5}{6}\right)^9 \cdot \left(\frac{\cancel{15}^{\,5}}{\cancel{6}_{\,2}}\right)$$

$$= \frac{5^{10}}{2 \times 6^9}$$

$${}^{10}C_0 = \frac{\cancel{10!}}{\cancel{10!}\,0!} = \frac{1}{1}$$

$${}^{10}C_1 = \frac{10!}{9!\,1!} = \frac{10 \cdot \cancel{9!}}{\cancel{9!} \times 1} = 10$$

Q.7 A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.



die $= \{1, 2, 3, 4, 5, 6\}$

$\frac{1}{6}$

(we need 2 success (getting '6') in first 5 throws) & (we need 1 success at 6th throw)

$\cap$

$P_2 = \frac{1}{6}$ —— (2)

$\boxed{n = 5}$, $\boxed{x = 2}$

$p = \frac{1}{6}$, $q = \frac{5}{6}$

Binomial Distribution, $P(x) = {}^nC_x \, p^x \cdot q^{n-x}$

$$P_1 = P(x = 2) = {}^5C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 \quad \text{—— (1)}$$

Required prob. $= P_1 \times P_2$

$$= {}^5C_2 \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$$

$$= \cancel{10}^{\,5} \cdot \frac{1}{\cancel{6}_{\,3}} \cdot \frac{5^3}{6^5} = \frac{5^4}{3 \times 6^5}$$

$$= \frac{625}{3 \times 7776} = \frac{625}{23328}$$

$${}^5C_2 = \frac{5!}{3!\,2!} = \frac{5 \cdot \cancel{4}^2 \cdot \cancel{3!}}{\cancel{3!}\,\cancel{2} \times 1} = 10$$

$$6^5 = 6^3 \cdot 6^2 = 216 \cdot 36$$

$$\begin{array}{r} 216 \\ \times\ 36 \\ \hline 1296 \\ 648\times \\ \hline 7776 \end{array}$$

Q.8 If a leap year is selected at random, what is the probability (chance) that it will contain 53 tuesdays?

Ans. No. of Days in a leap year = 366 Days.

$$7 \overline{)366}(52$$
$$-35$$
$$16$$
$$-14$$
$$2$$

$= \underbrace{7 \times 52}_{} + 2$

(Complete 52 weeks) → 52 tuesdays

Extra 2 Days

S-M
$\boxed{\text{M-T}}$
$\boxed{\text{T-W}}$
W-Th
Th-F
F-Sat.
Sat.-S

total 7-Case

$$P(53 \text{ tuesdays}) = \frac{2}{7}$$

Q.9 An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be at least 4 successes.

(Success) = 2 (fail)

$p = 2 \cdot q$ — ①

$\boxed{p + q = 1}$ — ②

$\Rightarrow 2q + q = 1$

$\Rightarrow 3q = 1$

$\Rightarrow \boxed{q = \frac{1}{3}}$ $\boxed{p = \frac{2}{3}}$

$n = 6$

no. of success $= x$

$\boxed{x \geq 4}$

$$P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$$

$$\boxed{P(x) = {}^{6}C_{x} \cdot \left(\frac{2}{3}\right)^{x} \cdot \left(\frac{1}{3}\right)^{n-x}}$$

$x \geq 4$, $x = 4, 5, 6$

ATQ

$$P(x \geq 4) = P(4) + P(5) + P(6)$$

$$P(x) = {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$$

$${}^nC_n = 1 = {}^nC_0$$

$${}^nC_{n-1} = n = {}^nC_1$$

$$P(x \geq 4) = P(4) + P(5) + P(6)$$

$$= \underbrace{{}^6C_4}_{15}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + \underbrace{{}^6C_5}_{6}\left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^1 + \overbrace{{}^6C_6}^{1}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0$$

$$= \left(\frac{2}{3}\right)^4 \cdot \left\{ \frac{15}{9} + 6 \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{4}{9} \right\}$$

$$= \left(\frac{2}{3}\right)^4 \cdot \left\{ \frac{15 + 12 + 4}{9} \right\}$$

$$= \left(\frac{2}{3}\right)^4 \left\{ \frac{31}{9} \right\}$$

$${}^6C_4 = \frac{6!}{2!\,4!} = \frac{\cancel{6}^{3} \cdot 5 \cdot \cancel{4!}}{\cancel{2} \times 1 \cdot \cancel{4!}} = 15$$

Q.10 How many times (n-times) must a man toss a fair coin so that the probability of having at least one head is more than 90%. ?

Success = getting 'H'

x = no. of success

$\boxed{x \geq 1}$

(Let a man tosses a coin - n-times)

H
T

$\boxed{P(x) = {}^nC_x\, p^x \cdot q^{n-x}}$

p = prob. of success $= \frac{1}{2}$

q = prob. of failure $= \frac{1}{2}$

$P(x \geq 1) = P(1) + P(2) + \dots + P(n)$

$= 1 - P(0)$

$= 1 - {}^nC_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{n-0}$

$= 1 - \left(\frac{1}{2}\right)^n$

$\boxed{P(x \geq 1) = 1 - \left(\frac{1}{2}\right)^n}$

ATQ, $P(x \geq 1) > \frac{9}{10}$

$\Rightarrow 1 - \frac{1}{2^n} > \frac{9}{10}$

$\Rightarrow 1 - \frac{9}{10} > \frac{1}{2^n}$

$\Rightarrow \frac{1}{10} > \frac{1}{2^n}$

ATQ

$P(x \geq 1) > 90\%$

$P(x \geq 1) > \frac{90}{100}$

$P(x \geq 1) > \frac{9}{10}$

H H H ... H (n-times)
T T T ... T

$x = 0, 1, 2, \dots n$?

all possible values of x

$\Rightarrow 10 < 2^n$

$\Rightarrow \boxed{2^n > 10}$

$n=1$ $2^1 > 10$ ✗

$n=2$ $2^2 > 10$ ✗

$n=3$ $2^3 > 10$ ✗

$n=4$ $2^4 > 10$ ✓

$n=5$ $2^5 > 10$ ✓

$n = 4, 5 \dots$

$\boxed{n \geq 4}$

Miscellaneous Exercise on Chapter 13

Q.11 In a game, a man wins a rupee for a six and losses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the mean expected value of the amount he wins / loses.

$\rightarrow \{\underbrace{1, 2, 3, 4, 5}_{\downarrow \ (-1₹)}, \underbrace{6}_{\downarrow \ (+1₹)}\}$

Random Variable
X = the amount he wins / loses

Success = getting a six.
Failure = ~~gent~~ not getting '6'

Case-I He gets '6' on 1st throw

$\boxed{6}$ $P(S) = \frac{1}{6}$

$X = +1$

Case-II. He gets '6' on 2nd throw.

$\boxed{\cancel{\times} \mid 6}$ $P(FS) = P(F \cap S) = P(F) \cdot P(S)$

$$= \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$$

$\boxed{X = -1 + 1 = 0}$

Case-III He gets '6' on '3rd' throw.

$\boxed{\times \mid \times \mid 6}$ $P(FFS) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216}$

$X = -1 - 1 + 1 = -1$

Case - IV (He does not get any '6' in any of the '3' throws)

x	x	x

$P(FFF) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}$

$X = -1 -1 -1 = -3$

Probability Distribution

X	1	0	-1	-3
$P(X)$	$1/6$	$5/36$	$25/216$	$125/216$

Expected value of $X = E(X) = \sum p_i x_i$

$= \frac{1}{6} \cdot (1) + \frac{5}{36} \cdot (0) + \left(\frac{25}{216}\right)(-1) + \left(\frac{125}{216}\right)(-3)$

$= \frac{1}{6} - \frac{25}{216} - \frac{375}{216} = \frac{36 - 25 - 375}{216}$

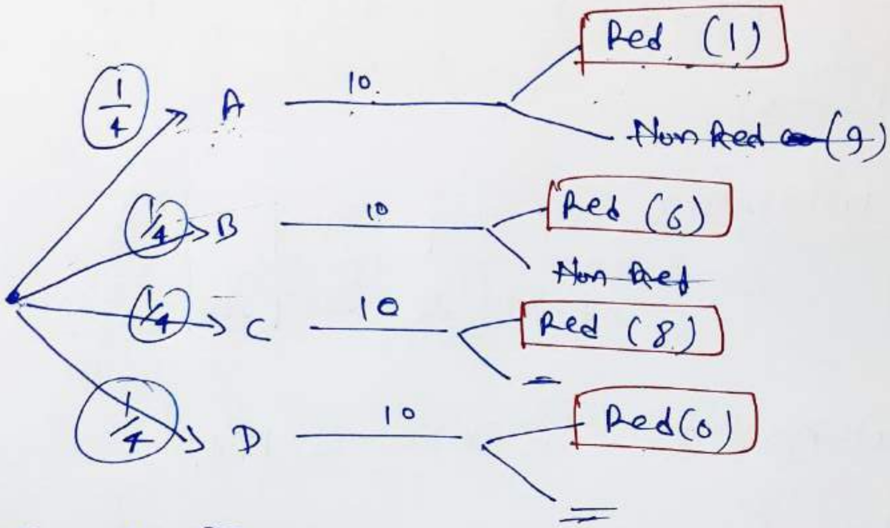
$= \frac{-\cancel{364}\ 91}{\cancel{216}\ 54} = -\frac{91}{54}$

Q. 12

	Red	White	Black	Total
A	1	6	3	→ 10
B	6	2	2	→ 10
C	8	1	1	→ 10
D	0	6	4	→ 10

Bayes' Theorem

A E P B



E_1 : Box A

E_2 : Box B

E_3 : Box C

E_4 : Box D

R : getting Red marble.

$P(E_1) = \frac{1}{4}$	$P(R \mid E_1) = 1/10$
$P(E_2) = \frac{1}{4}$	$P(R \mid E_2) = \frac{6}{10}$
$P(E_3) = \frac{1}{4}$	$P(R \mid E_3) = 8/10$
$P(E_4) = \frac{1}{4}$	$P(R \mid E_4) = 0/10$

~~R: getting Red marble~~

$$P(E_1 \mid R) = \frac{P(E_1) \cdot P(R \mid E_1)}{\left(P(E_1) \cdot P(R \mid E_1) + P(E_2) \cdot P(R \mid E_2) + P(E_3) \cdot P(R \mid E_3) + P(E_4) \cdot P(R \mid E_4) \right)}$$

$$P(E_1|R) = \frac{\left(\cancel{\tfrac{1}{4}} \cdot \tfrac{1}{\cancel{10}}\right)}{\left(\cancel{\tfrac{1}{4}} \cdot \tfrac{1}{\cancel{10}}\right) + \left(\cancel{\tfrac{1}{4}} \cdot \tfrac{6}{\cancel{10}}\right) + \left(\cancel{\tfrac{1}{4}} \cdot \tfrac{8}{\cancel{10}}\right) + \left(\cancel{\tfrac{1}{4}} \cdot \tfrac{0}{\cancel{10}}\right)}$$

$$= \frac{1}{1+6+8+0} = \frac{1}{15} \checkmark$$

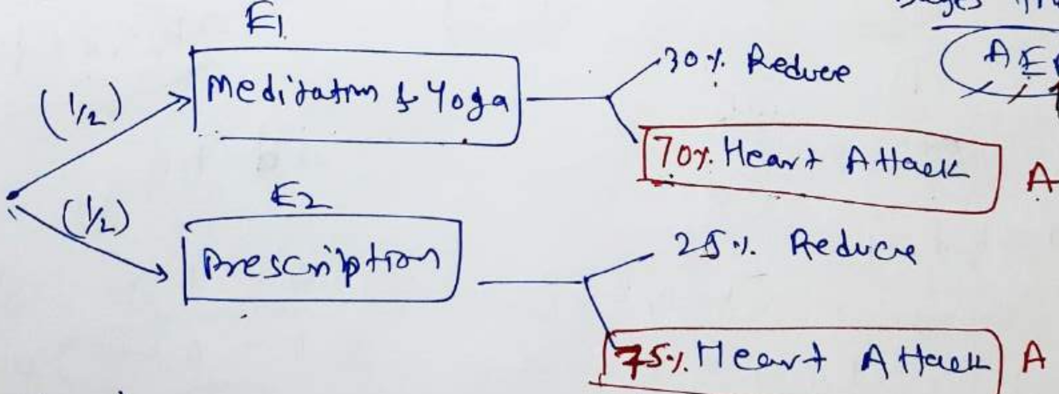
Similarly

$$P(E_2|R) = \frac{6}{1+6+8+0} = \frac{6}{15} = \frac{2}{5} \checkmark$$

$$P(E_3|R) = \frac{8}{1+6+8+0} = \frac{8}{15} \checkmark$$

$\boxed{\text{Q. 13}}$

Bayes' theorem



$P(E_1) = \frac{1}{2}$ — $P(A|E_1) = 70\% = \frac{70}{100}$

$P(E_2) = \frac{1}{2}$ — $P(A|E_2) = 75\% = \frac{75}{100}$

$$P(\text{Med}^n\text{. \& Yoda} \,|\, A) = P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

(Bayes' theorem)

$$= \frac{\left(\cancel{\tfrac{1}{2}} \times \tfrac{70}{\cancel{100}}\right)}{\left(\cancel{\tfrac{1}{2}} \times \tfrac{70}{\cancel{100}}\right) + \left(\cancel{\tfrac{1}{2}} \times \tfrac{75}{\cancel{100}}\right)} = \frac{70}{145} = \frac{14}{29} \checkmark$$

Q. 14

Let determinant of second order (2×2) =

$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\quad a, b, c, d \in \{0, 1\}$.

total no. of Determinants $= 2 \cdot 2 \cdot 2 \cdot 2 = 16$

2 ways → 2 ways

2×2

2 ways   2 ways.

Determinant = +ve

$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

+ve.

$\Delta > 0$

$\Delta = \underbrace{ad}_{1} - \underbrace{bc}_{0/1} = +ve.$

$a, b, c, d \in \{0, 1\}$

$a = 1, d = 1$ fixed.

$bc = 0$

$b = 0, c = 1$

$b = 1, c = 0$

$b = 0, c = 0$

$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

ad	− bc	
0	− 0	✗
0	− 1	✗
1	− 0	✓ = +
1	− 1	✗

P(Determinant value is positive) $= \dfrac{3}{16}$

Q.15

$P(A \text{ fails}) = 0.2 = P(A)$

$P(B \text{ fails alone}) = 0.15 = P(B-A)$

$P(A \text{ and } B \text{ fails}) = 0.15 = P(A \cap B)$

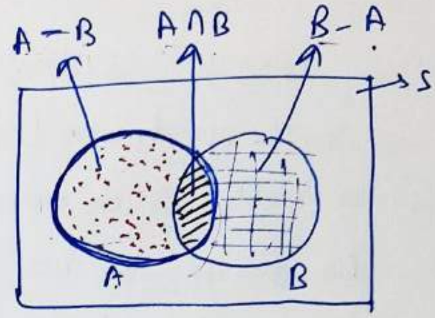
(i) $P(A \text{ Fails} \mid B \text{ has failed}) = ?$

(ii) $P(A \text{ fails alone}) = ?$

Event

A = A fails

B = B fails



→ A fails Alone

→ A & B fail

→ B fails alone

(i) $P(A \text{ fails} \mid B \text{ has failed})$

$= P(A \mid B)$

$= \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.15}{0.30} = \dfrac{15}{30}$

$= \dfrac{1}{2} = 0.5$

(ii) $P(A \text{ fails alone})$

$= P(A-B)$

$= P(A) - P(A \cap B)$

$= 0.20 - 0.15$

$= 0.05$

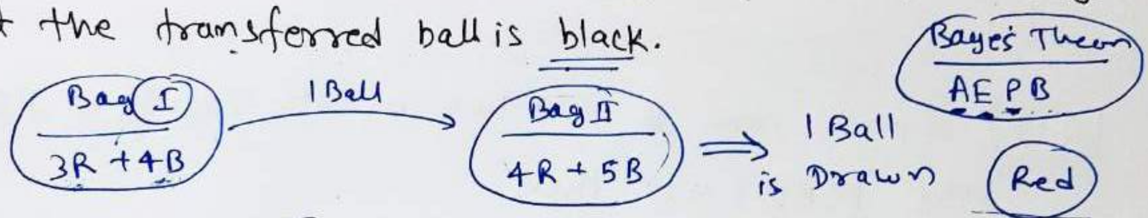
$P(B) = P(A \cap B) + P(B-A)$

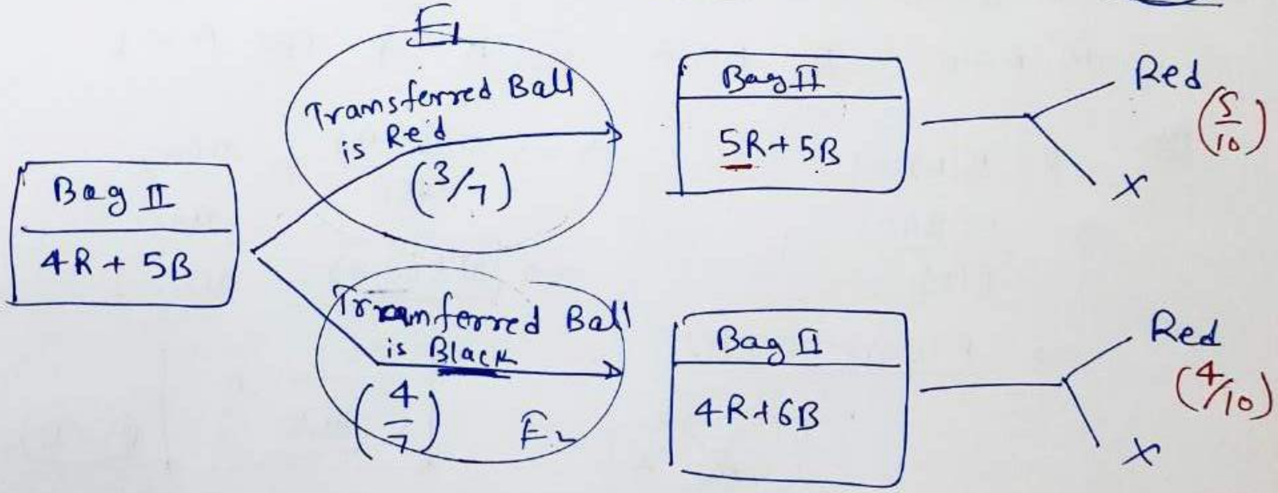
$= (0.15) + 0.15$

$= 0.30$

Miscellaneous Exercise on Chapter (13)

Q.16 Bag I contains 3 red & 4 black balls and Bag II contains 4 red & 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.





E_1 = Transferred ball is Red

E_2 = Transferred ball is Black.

A: Drawing a Red colour Ball from Bag II.

$P(E_1) = 3/7$	$P(A\|E_1) = \frac{5}{10}$
$P(E_2) = 4/7$	$P(A\|E_2) = \frac{4}{10}$

ATQ. P (transferred Ball is Black (E_2) given that Red colour Ball is drawn from Bag II (A)) $= P(E_2|A)$

$$P(E_2|A) = \frac{P(E_2) \cdot P(A|E_2)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\left(\frac{4}{7} \times \frac{4}{10}\right)}{\left(\frac{3}{7} \cdot \frac{5}{10}\right) + \left(\frac{4}{7} \cdot \frac{4}{10}\right)} = \frac{16}{15+16} = \frac{16}{31}$$

$\boxed{Q.17}$ If A and B are two events such that $P(A) \neq 0$ and $P(B|A) = 1$, then

(A) $A \subset B$ (B) $B \subset A$ (C) $B = \phi$ (D) $A = \phi$

Ans

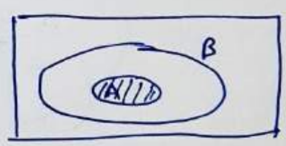
$P(B|A) = 1$

$\Rightarrow \frac{P(B \cap A)}{P(A)} = 1$

$\Rightarrow P(B \cap A) = P(A)$

$\Rightarrow \frac{n(B \cap A)}{\cancel{n(S)}} = \frac{n(A)}{\cancel{n(S)}}$

$\Rightarrow \boxed{n(B \cap A) = n(A)}$



$A \subset B$

$A \cap B = A$

$\boxed{Q.18}$ If $P(A|B) > P(A)$, then which of the following is correct:

(A) $P(B|A) < P(B)$ (B) $P(A \cap B) < P(A) \cdot P(B)$

(C) $P(B|A) > P(B)$ (D) $P(B|A) = P(B)$

$P(A|B) > P(A)$

$\Rightarrow \frac{P(A \cap B)}{P(B)} > P(A)$

$\Rightarrow P(A \cap B) > P(A) \cdot P(B)$

$\Rightarrow \frac{P(A \cap B)}{P(A)} > P(B)$

$\Rightarrow \boxed{P(B|A) > P(B)}$

Q.19 If A and B are any two events such that $P(A) + P(B) - P(A \text{ and } B) = P(A)$, then

(A) $P(B|A) = 1$ (B) $P(A|B) = 1$

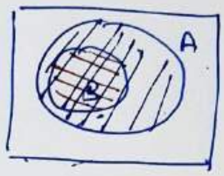
(C) $P(B|A) = 0$ (D) $P(A|B) = 0$

$$P(A) + P(B) - P(A \cap B) = P(A)$$

$$\Rightarrow \boxed{P(A \cup B) = P(A)}$$

$$\Rightarrow \frac{n(A \cup B)}{\cancel{n(S)}} = \frac{n(A)}{\cancel{n(S)}}$$

$$n(A \cup B) = n(A)$$



$A \cup B = A$

$\cancel{A \cup B = B}$

$B \subset A$

$\boxed{A \cap B = B}$

A/C $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(B)}{P(A)}$

B/D $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\cancel{P(B)}}{\cancel{P(B)}} = 1$

The End