Relations and Functions (संबंध एवं फलन)

Set $A = \{a, b, c\}$ = set of cities

a → Ajmer
b → Bombay
c → Chennai

Set $B = \{r, s, t\}$ = set of states

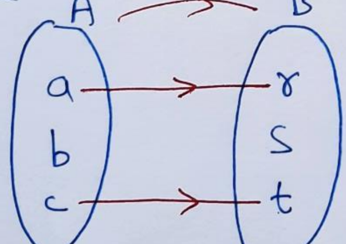
r → Rajasthan
s → Sikkim
t → Tamilnadu

Cartesian Product (कार्तीय गुणन) $A \times B = \{(a, r), (a, s), (a, t), (b, r), (b, s), (b, t), (c, r), (c, s), (c, t)\}$

'set of all odered pairs from A to B'

(Cities) A → B (States)

from A to B



← Arrow Diagram

Set Builder Form

Relation 'R_1' $= \{(x, y) : x \in A, y \in B$, city x lies in state $y\}$

Relation 'R_1' $= \{(a, r), (c, t)\}$ ← Roster Form

Relation is a subset of Cartesian Product $A \times B$: from A to B $\boxed{R \subseteq A \times B}$

$(a, r) \in R_1$ या $a R_1 r$

$(c, t) \in R_1$ या $c R_1 t$

$(a, s) \notin R_1$

Relation A to A ⇓ Relation on 'A' (in)

Empty Relation (रिक्त संबंध) $\phi \subseteq A \times A$

set $A = \{1, 2, 3\}$

set $B = \{100, 101, 102\}$

Relation $R_2 = \{(a, b) : a \in A,\ b \in B,\ \underline{a > b}\} = \{\ \} = \phi$

impossible.

Universal Relation (सार्वत्रिक संबंध) $= A \times A$

$$R_3 = \{(a, b) : a \in A,\ b \in B,\ |a - b| \geq 0\} = \left\{\begin{matrix}(1,100)\\(1,101)\\ \vdots \\ \vdots \\(3,102)\end{matrix}\right\}$$

$\{1, 2, 3\}$ $\qquad$ $\{100, 101, 102\}$

100% Possible (Sure) $\qquad$ All

$|1 - 100| = |-99| \geq 0$

$= +99 \geq 0$ True

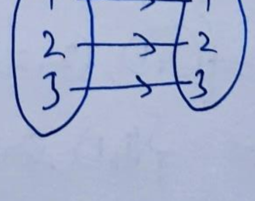
Different Types of relations on 'A'

Relation 'R' from A to A $\Rightarrow$ R on 'A'

Extra

I Identity Relation:

$A = \{1, 2, 3\} \longrightarrow A = \{1, 2, 3\}$



$I_A = \{(1,1), (2,2), (3,3)\}$ ✓

$\times$ $R_1 = \{(1,1), (2,2)\}$ $\qquad$ $R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\}$ $\times$

II Reflexive (R) Relations (on A)

A to A

$\{1,2,3\}$ $\{1,2,3\}$

$\boxed{\text{if } (a,a) \in R \ \forall \ a \in A}$

(for every)

$A = \{1,2,3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ Reflexive ✓ Identity ✓

$R_2 = \{(1,1), (2,2)\}$ Reflexive ✗ Identity ✗

$R_3 = \{(1,1), (2,2), (3,3), (1,2)\}$ Reflexive ✓ Identity ✗

$R_4 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (3,1)\}$ Reflexive ✓ Id. ✗

III Symmetric Relations (समामित संबंध)

A ⟶ A

$\{1,2,3\}$ $\{1,2,3\}$

$R \Rightarrow$ if $(a,b) \in R$ then $(b,a) \in R$

$R_1 = \{(1,1), (2,2)\}$ ✓ $(1,1) \in R_1$, $(1,1) \in R_1$

$R_2 = \{(1,1), (2,2), (3,3)\}$ ✓

$R_3 = \{(1,1), (2,2), (3,3), (1,2)\}$ ✗

$R_4 = \{(1,2), (2,1), (3,2)\}$ Sym. ✗

$R_5 = \{(1,2), (2,1)\}$ Sym. ✓

(IV) Transitive Relations (संक्रामक संबंध)

if $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$

$\Rightarrow$

Transitive ✓ ✗

$A = \{1,2,3\} \longrightarrow A = \{1,2,3\}$

✓ $R_1 = \{(1,2),(2,3),(1,3)\}$ $(1,2) \in R_1, (2,3) \in R_1 \Rightarrow (1,3) \in R_1$

✓ $R_2 = \{(1,1)\}$ $(1,1) \in R_2$ $(1,1) \in R_2 \Rightarrow (1,1) \in R_2$

✓ $R_3 = \{(1,1)(2,2),(3,3)\}$

☆ $R_4 = \{(1,2)\}$ ✓

$\boxed{\text{If rain then play}}$

✓ $R_5 = \{(1,1),(2,2),(3,3),(1,2)\}$

$(1,1) \longrightarrow (1,1) \xrightarrow{\text{then}} (1,1)$

$\longrightarrow (1,2) \xrightarrow[\text{then}]{} (1,2)$

✗ $R_6 = \{(1,2),(2,3),(3,1)\}$

$(1,3) \notin R_6$

(V) Equivalence Relation (तुल्यता संबंध)

Reflexive $(a,a) \in R \ \forall\ a \in A$

Symmetric $(a,b) \Rightarrow (b,a)$

Transitive $(a,b), (b,c) \Rightarrow (a,c)$

e.g. $A = \{1, 2, 3\}$ $R : A \to A$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$

(I) Reflexive $(a,a) \in R \ \forall a \in A$ ✓

(II) Symmetric ✓

$(a,b) \Rightarrow (b,a)$

(III) Transitive ✓

If $(a,b) \in R, (b,c) \in R$

then $(a,c) \in R$

Equivalence Relation

Exercise 1.1 [Relations and Functions]

R ~~for~~ from 'A' to 'B'

Reflexive Relations $\boxed{\text{in 'A'} \text{ on 'A'}} = \boxed{\text{from A to A}}$

$(a,a) \in R$ for every $a \in A$.

Equivalence Relation

Symmetric Relations

if $(a,b) \in R$ then $(b,a) \in R$.

Transitive Relations

If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

T ✓

T ✓

T ✗

Exercise 1.1

$\boxed{\text{Q.1}}$ Reflexive / Symmetric / Transitive

(i) $A = \{1, 2, 3, --- 13, 14\}$ (in A) $x \in A$, $y \in A$

$R = \{(x,y) : 3x - y = 0\}$

Reflexive, for (x,x) $3x - x = 0 \Rightarrow 2x = 0 \Rightarrow \boxed{x = 0} \notin A$

$(0,0)$ $x \notin A$

Not reflexive.

Symmetric If $\underbrace{(x,y) \in R}_{3x - y = 0}$ then $\underbrace{(y,x) \notin R}_{3y - x = 0}$

Not Sym.

$\not\Rightarrow$

Transitive,

If $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \notin R$

$3x - y = 0$ — (1) $\quad$ $3y - z = 0$ — (2) $\quad$ $\boxed{3x - z = 0}$??

$3x = y$ $\Rightarrow 3(3x) - z = 0$

$\boxed{9x - z = 0}$

Not Transitive

(ii) Relation in N (set of natural no.)

(from N to N) $\{1, 2, 3 \ldots\}$

$R = \{(x, y) : \underbrace{y = x + 5 \text{ and } x < 4}\}$

Set Builder Form

$x = 1, 2, 3 \quad y = 6, 7, 8$

$R = \{(1,6), (2,7), (3,8)\}$

Reflexive

(1,1) (2,2) (3,3) (4,4) - - - ×

Not reflexive

Symmetric

$(1,6) \in R$ but $(6,1) \notin R$

$\therefore$ Not sym.

Transitive,

$(1,6) \in R$

$(2,7) \in R$

$(3,8) \in R$

only

Transitive,

(iii) Relation in the set $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) : y \text{ is divisible by } x\}$

Reflexive. $(x, x) \in R \quad \forall x \in A$

$\downarrow$

x is divisible by $x \quad \forall x \in A$

(True)

Yes. Reflexive ✓

Symmetric: If $(x, y) \in R$ then $(y, x) \in R$.

y is divisible by x

$6 \longrightarrow 2$ ✓

'x' is divisible by 'y'

$2 \longrightarrow 6$ ✗

Not sym.

Transitive. If $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

y is divisible by x

$y = kx$

z is divisible by y

$z = m \cdot y$

$\Rightarrow z = m k x$

z is divi. by x

$z = \boxed{mk} \cdot x$

Yes Transitive

(iv) Relation in set $Z \leftarrow$ (all integers)

$\{\ldots -2, -1, 0, 1, 2, 3 \ldots\}$

$R = \{(x, y) : x - y \text{ is an integer}\}$

Reflexive: $(x, x) \in R \quad \forall\, x \in Z$

$\forall$ → for every

yes ✓

$\underbrace{x - x}_{0}$ is an integer $\forall\, x \in Z$

True ✓

Symmetric: If $(x, y) \in R$ then $(y, x) \in R$

yes

$(x-y)$ is an integer

$(x-y) = I$

$\Rightarrow -(y-x) = I$

$\Rightarrow (y-x) = -I$

$(y-x)$ is an integer

Transitive: If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

yes

$(a-b)$ is an integer & $(b-c)$ is an integer

$(a-c)$ is an integer

$a - \cancel{b} = I_1$

$\cancel{b} - c = I_2$

$+$

$a - c = I_1 + I_2 = \text{integer}$ ✓

(v) relation R in set 'A' of human beings in a town.

(a) $R = \{(x,y) : x$ and y work at the same place$\}$

(b) $R = \{(x,y) : x$ and y live in the same locality$\}$

Reflexive	Symmetric	Transitive
$(x,x) \in R$ ✓	$(x,y) \in R \Rightarrow (y,x) \in R$ ✓ ✓	$(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$ ✓

(c) $R = \{(x,y) : x$ is exactly 7cm taller than $y\}$

Reflexive	Sym	Transitive
$(x,x) \notin R$ Not	$(x,y) \in R \Rightarrow (y,x) \notin R$ ✗ Not	$(x,y) \in R$ & $(y,z) \in R \Rightarrow \cancel{(x,z)} \notin R$

$x = z + 7$

$x = y + 7$ ①

$y = z + 7$ ②

$x = (z+7) + 7$

$x = z + 14$

Not

(d) $R = \{(x,y) : x$ is wife of $y\}$

Reflexive	Sym.	Transitive
$(x,x) \in R$ ✗ Not	$(x,y) \in R \Rightarrow (y,x) \in R$	$(x,y) \in R, (y,z) \notin R \Rightarrow (x,z) \in R$
	W H ✓ ✗ W H ?	W H ~~W H~~
		Transitive

(e) $R = \{(x,y) : x \text{ is father of } y\}$

Reflexive

$(x,x) \in R$

✗

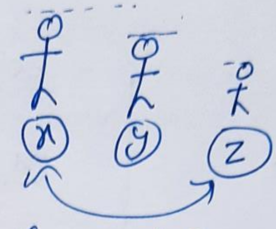
Sym.

$(x,y) \in R \Rightarrow (y,x) \in R$

✗

Transitive

$\underbrace{(x,y) \in R,\ (y,z) \in R}$ $\Rightarrow$ $\underbrace{(x,z) \notin R}_{\text{Father}}$



$\nRightarrow$

Grand Father

✗

Exercise 1.1 (Relations and Functions)

[Q.2] Relation in the set 'R' ← (real numbers)

$R = \{(a,b) : a \leq b^2\}$ → Reflexive ✗, Symmetric ✗, Transitive ✗ } Prove

Reflexive

real no. ↓ $-\infty \quad 0 \quad 1 \quad \infty$

$(x,x) \in R \ \forall x \in R$ (relation ↑, for every ↓) $\Rightarrow x \leq x^2$

$-1, -2$
$1, 2, 3, 4, \ldots$ ✓
0 ✓

$x = 0.5 = \frac{1}{2}$

$\frac{1}{2} \nleq \left(\frac{1}{2}\right)^2$

$\frac{1}{2} \nleq \frac{1}{4}$

$0.5 > 0.25$

this is not true for $x \in (0,1)$

Not reflexive.

Symmetric If $(x,y) \in R$ then $(y,x) \in R$

$x \leq y^2$ → $y \leq x^2$

$x = 1,\ y = 2$

$1 \leq 4$ ✓ $\qquad$ $2 \leq 1$ ✗

Not Sym.

Transitive If $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$

Not

$a \leq b^2$ ✓ $\qquad b \leq c^2$ ✓ } $\qquad a \leq c^2$

$a = 7,\ b = -5 \qquad c = 1$

$7 \leq 25$ ✓ $\qquad -5 \leq 1$ ✓ $\qquad \boxed{7 \leq 1 \ ✗}$

Q.3 Relation (R) in the set $A = \{1, 2, 3, 4, 5, 6\}$

a, b

$R = \{(a,b) : b = a+1\}$

Reflexive.

$(a,a) \in R \ \forall a \in A$

$a = a+1$

$\Rightarrow 0 = 1$

False

Not Reflexive

Symmetric

$(a,b) \in R \Rightarrow (b,a) \in R$

$b = a+1 \qquad a = b+1$

✗

Not

Transitive

$(a,b) \in R,\ (b,c) \in R$

$\Rightarrow (a,c) \notin R$

$b = a+1$ — (1)

$c = b+1$ — (2)

$c = a+2$

$(a,c) \rightarrow c = a+1$

Not

Q.4 Relation 'R' in set 'R' ← Real no.

$R = \{(a,b) : a \le b\}$ → Reflexive ✓, Transitive ✓, Sym. ✗

Reflexive.

$(a,a) \in R$

$a \le a$

✓

✓

Sym.

$(a,b) \in R \Rightarrow (b,a) \in R$

$a \le b \qquad b \le a$

$2 \le 5 \qquad 5 \le 2$

✗

Not sym.

Transitive.

$(a,b) \in R,\ (b,c) \in R$

$\boxed{a \le b} \qquad b \le c$

$\Downarrow$

$(a,c) \in R$

a b c

$a \le c$ ✓

Transitive ✓

Q. 5 relation R in $\mathbb{R}$ → real no.

$R = \{(a,b) : a \le b^3\}$

$-\infty \quad 0 \quad 1 \quad \infty$

Reflexive

$(a,a) \in R \;\forall a \in R$

$a \nleq a^3$

$a = \frac{1}{2}$

$\frac{1}{2} \nleq \left(\frac{1}{2}\right)^3$

$\frac{1}{2} \nleq \frac{1}{8}$

$0.5 > 0.125$

Not

Transitive

$(a,b) \in R$, $(b,c) \in R$

$a \le b^3$ ⇐ $b \le c^3$

⇓

$(a,c) \notin R$

$a \nleq c^3$

$a \le (b)^3 \le (c^3)^3$

$a \le c^9$

Not transitive

Symmetric

$(a,b) \in R \Rightarrow (b,a) \in R$

$a \le b^3$ → $b \nleq a^3$ ✗

$a = 1$

$b = 2$

Not Sym.

Q.6 Relation R in the set $\{1,2,3\}$

$R = \{(1,2), (2,1)\}$
→ Sym. ✓
→ Reflexive ✗
→ Trans. ✗

Reflexive

$\{(1,1)\,(2,2),(3,3)\}$

✗

Sym.

$(1,2) \in R$ then $(2,1) \in R$

✓

Transitive, $(a,b),(b,c) \Rightarrow (a,c)$

$(1,2)\;(2,1) \Rightarrow (1,1) \notin R$

Not transitive

Q.7 in set 'A' of all books in a library.

$$R = \left\{ (x, y) : x \text{ and } y \text{ have same no. of pages} \right\}$$

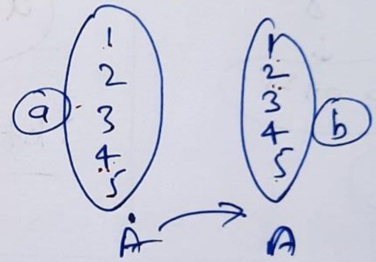
✓

Equivalence

- Reflexive → (x, x) → ✓
- Symmetric → $(x, y) \in R \Rightarrow (y, x) \in R$ ✓
- Transitive → $(x, y) \in R$, $(y, z) \in R \Rightarrow (x, z) \in R$ ✓

Q.8 Relation in set $A = \{1, 2, 3, 4, 5\}$

$$R = \{(a, b) : |a - b| \text{ is even}\}$$



$R = \{ (1,1), (1,3), (1,5)$
$(2,2), (2,4)$
$(3,1), (3,3), (3,5)$
$(4,2), (4,4)$
$(5,1), (5,3), (5,5) \}$

$\{1, 3, 5\}$

$|\text{odd} - \text{odd}| = \text{even}$

$|\text{even} - \text{even}| = \text{even}$

Reflexive	Symmetric	Transitive
$(a, a) \in R \ \forall a \in A$	$(a, b) \Rightarrow (b, a)$	$(a, b), (b, c) \Rightarrow (a, c)$
✓	✓	✓

Q.9 Relation in the set $A = \{x \in \mathbb{Z} : 0 \le x \le 12\}$

$A = \{0, 1, 2, 3 \dots 12\}$

(i) $R = \{(a, b) : |a-b|$ is a multiple of $4\}$

(ii) $R = \{(a, b) : a = b\}$

(i) $R = \{(a, b) : |a-b|$ is a multiple of $4\}$

Reflexive. $(a, a) \in R \;\; \forall \; a \in A$

✓ $\Rightarrow |a-a| = 0$ is a multiple of 4 ✓

Sym. $(a, b) \in R \Rightarrow (b, a) \in R$

✓ $|a-b|$ is a multiple of 4 — $|b-a|$ is a multiple of 4

True.

Transitive. $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

✓ $|a-b| = 4k$ $\quad |b-c| = 4m$ $\quad |a-c| = 4n$

$a-b = 4k$ $\quad b-c = 4m$

(+)

~~Fran~~

$a-c = 4k + 4m$

$a-c = 4(k+m)$

$|a-c| = 4|k+m|$

elements related to 1 $\quad (a,b) \in R$

$a = 1$ $\quad b = ?$

(i) $|a-b|$ is a multiple of 4

0, 4, 8, 12, ...

$|1-b|$ is a multiple of 4

$b = 1, 5, 9$

(ii) $a = b$

(a, a) —

$(a, b)\ (b, a)$ —

$(a, b)\ (b, c)\ (a, c)$ —

$a = 1$ $\quad (a, b) \in R$

$1 = b$ $\quad 1$

$a = b$

Exercise 1.1 (Relations and Functions)

$\boxed{\text{Q11 to Q16}}$

$\boxed{\text{Q.11}}$ Relation in the set A of points in a plane.

$R = \{(P, Q) :$ distance of the point P from origin is same as the distance of the point Q from origin $\}$

$OP = OQ$

Reflexive

$(P, P) \in R \ \forall \ P \in A$

origin = 'O' $(0,0)$

$OP = OP$

True.

Reflexive

Symmetric

$(P, Q) \in R \Rightarrow (Q, P) \in R$

$OP = OQ \qquad OQ = OP$

Transitive.

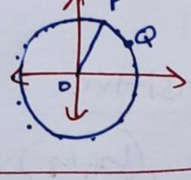
$(P, Q) \in R \ , (Q, S) \in R$

$OP = OQ$ — ① $\Downarrow$ $OQ = OS$ — ②

$(P, S) \in R$

$\boxed{OP = OS}$

Equivalence



Sides

$T_1 \rightarrow 3, 4, 5$

$T_2 \rightarrow 5, 12, 13$

$T_3 \rightarrow 6, 8, 10$

$T_1 \sim T_3$

$\boxed{\text{Q.12}}$ Relation in the set A of all triangles.

$R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\} \rightarrow T_1 \sim T_2$

Reflexive.	Symmetric	Transitive
$T_1 \sim T_1$	$T_1 \sim T_2 \Rightarrow T_2 \sim T_1$	$T_1 \sim T_2, \ T_2 \sim T_3 \Rightarrow \boxed{T_1 \sim T_3}$

Q.13 Relation in the set A of all polygons

$R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same no. of sides}\}$

$n(P_1) = n(P_2)$

let: $n(P)$ denotes no. of sides of 'P'

Reflexive.

$(P_1, P_1) \in R \ \forall \ P_1 \in A$

$n(P_1) = n(P_1)$

✓

Sym.

$(P_1, P_2) \in R \Rightarrow (P_2, P_1) \in R$

$n(P_1) = n(P_2) \quad n(P_2) = n(P_1)$

✓

Transitive $(P_1, P_2) \in R$, $(P_2, P_3) \in R \Rightarrow (P_1, P_3) \in R$

$n(P_1) = n(P_2)$ —(1), $n(P_2) = n(P_3)$ —(2)

$\therefore n(P_1) = n(P_3)$

✓

Q.14 Relation in set L of all lines in XY plane.

$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$

$L_1 \parallel L_2$

Reflexive

$(L_1, L_1) \in R$

$L_1 \parallel L_1$

✓

Sym.

$(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$

$L_1 \parallel L_2 \quad L_2 \parallel L_1$

✓

Transitive

$(L_1, L_2), (L_2, L_3) \Rightarrow (L_1, L_3)$

$L_1 \parallel L_2 \quad L_2 \parallel L_3 \quad L_1 \parallel L_3$

✓

$\xleftrightarrow{m} L_1$

$\xleftrightarrow{m} L_2$

$\xleftrightarrow{m} L_3$

$y = mx + c$

Slope

Equivalence ✓

$y = 2x + 4 \longrightarrow \boxed{y = 2x + c} \quad c \in R$

$\parallel$

Q.15 Relation in the set $\{1,2,3,4\} = A$

$R = \{(1,2), (2,2), (1,1), (4,4), (1,3), (3,3), (3,2)\}$

Reflexive	Symmetric	Transitive
$(a,a) \in R$ $\forall a \in A$ ✓	Here $(1,2) \in R$ but $(2,1) \notin R$ ✗	$(1,2) \in R$ $(2,2) \in R$ $(1,2) \in R$ ✓ $(a,b), (b,c) \Rightarrow (a,c)$ ✓

Option (B)

Q.16 Relation in the set N ✓

$R = \{(a,b) : \boxed{a = b - 2}, b > 6\}$

(A) $(2,4) \in R$ ✗ $4 < 6$

(B) $(3,8) \in R$ $3 \neq 8-2$

(C) $(6,8) \in R$ $6 = 8-2$ ✓

(D) $(8,7) \in R$ $8 \neq 7-2$

Option (C)

→ One-one Function (Injective Fn.) (एकैकी फलन)

→ many-one Function (बहुएकैकी फलन)

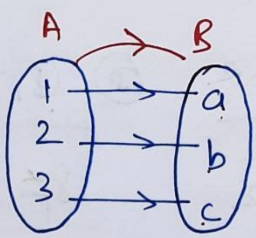
→ Onto Function (Surjective fn.) (आच्छादक फलन)

→ Into Function (अन्तर्क्षेपी फलन)

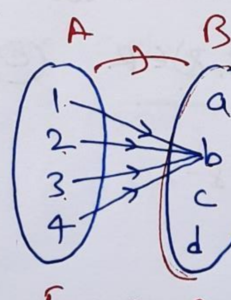
⇒ One One Onto Fn. (Bijective Fn.)

Function (फलन)

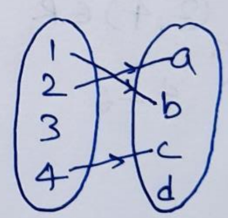
Function is a special type of relation from a non empty set A to another non empty set B such that each element of set A has unique image in set B.



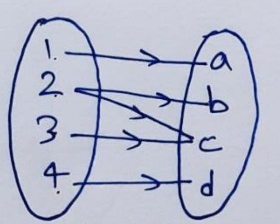
Function



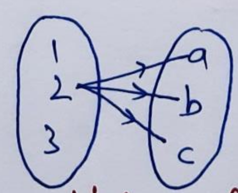
Function.



Not a function



Not a function



Not a func.

image of '1' = a
Preimage of 'a' = 1

function $f : A \to B$

Domain — input (x)

Codomain

$\boxed{\text{Range}}$ = Set of images

output ($y = f(x)$)

$\boxed{\text{Range} \subseteq \text{Codomain}}$

One One Functions (Injective Fn^n.) (एकैकी फलन)

$f : X \to Y$

images of distinct elements of X under 'f' are distinct

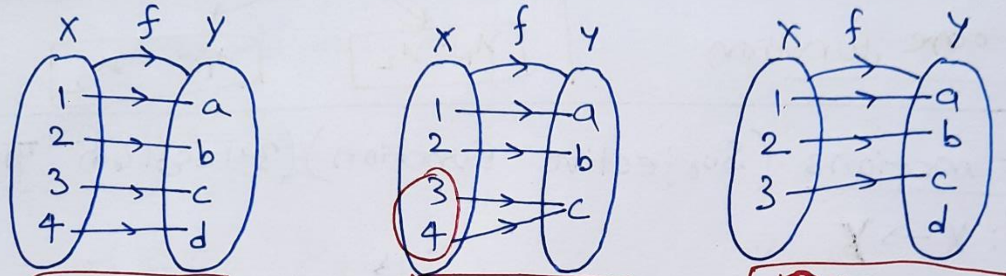
$x_1, x_2 \in X$

only

$$\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$$

$$\boxed{x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)}$$

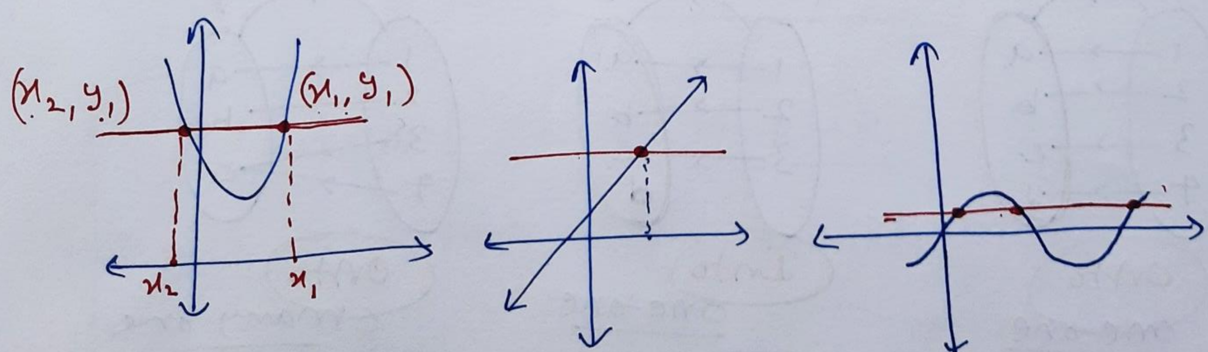
Many-one Functions : which are not one-one.



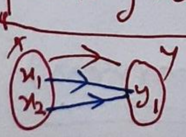
one-one | many-one | one-one

Horizontal line Test : (for graphs)

If a Horizontal line cuts the graph of a function at atmost (ज्यादा से ज्यादा) one point, then this graph represents a one-one function.



many-one | one-one | many-one Fn^n.



e.g. Check injectivity (one-one / many-one)

(i) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 3x + 7$

let $x_1, x_2 \in \mathbb{R}$

$f(x_1) = f(x_2)$

$\Rightarrow 3x_1 + \cancel{7} = 3x_2 + \cancel{7}$

$\Rightarrow \boxed{x_1 = x_2}$ only.

one-one function

(ii) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

let $x_1, x_2 \in \mathbb{R}$

$f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow \boxed{x_1 = \pm x_2}$ many-one

$\boxed{x_1 = x_2}$ $\boxed{x_1 = -x_2}$

Onto Functions (Surjective Function) (आच्छादक फलन)

(पर) $f: X \to Y$

if every element of Y, is the image of some element of X under f.

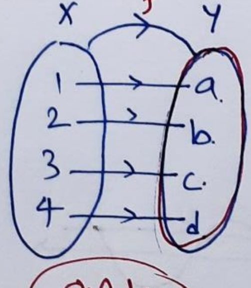
$\boxed{\text{Range} = \text{Codomain}(Y)}$

Connected in Y

(में)

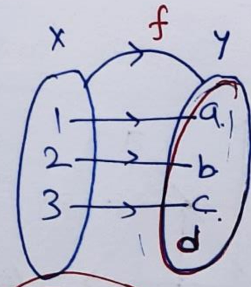
Into Functions : which are not onto.

e.g.



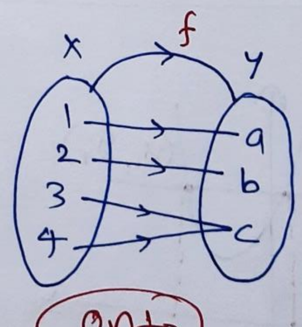
onto

one-one



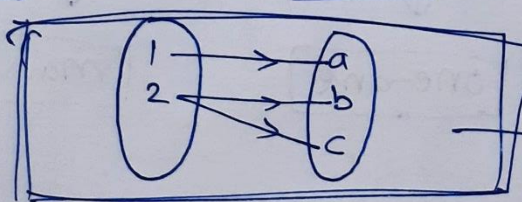
Into

one-one



onto

many-one



Comment

Bijective Functions (one-one Onto Functions)

which are one one & onto both.

$f(x_1)=f(x_2) \rightarrow x_1=x_2$

Range = codomain

e.g. Check Surjectivity (Hence Bijectivity)

onto / into

(i) $f(x) = 3x+7$

$f: R \rightarrow R$

Codomain

$x \in R \leftarrow$ Domain

$3x+7 \in R$

$f(x) \in R \leftarrow$ Range.

output Range = R = Codomain

onto ✓

one-one – Bijective

(ii) $f(x) = x^2$

$f: R \rightarrow R$

Codomain.

Range.

$x \in R$

$x^2 \geq 0$ Square ≥ 0

$x^2 \in [0,\infty)$

$f(x) \in [0,\infty) \leftarrow$ Range.

output

Range $= [0,\infty) \neq R$ $(-\infty,\infty)$

into many-one

Tip ☆ Questions with $f: N \rightarrow N$

Draw Arrow Diagram

1, 2, 3. . .

☆ Questions with $f: R \rightarrow R$

Apply Proper methods

$f(x_1) = f(x_2)$ (one-one) $\rightarrow x_1 = x_2$

Range = Codomain (onto)

Exercise 1.2 [Relations and Functions]

Q.1 $f : \underbrace{R_*}_{\text{Domain}} \to \underbrace{R_*}_{\text{Codomain}}$ defined by $f(x) = \frac{1}{x}$

$R - \{0\} = (-\infty, 0) \cup (0, \infty)$

$-\infty \quad 0 \quad \infty$

One-one let $f(x_1) = f(x_2)$ Prove $\boxed{x_1 = x_2}$ only

$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$

$\Rightarrow x_1 = x_2$ only $\quad \therefore f \to$ one-one

onto $\boxed{\text{Range} = \text{codomain } (R_*)}$ Prove

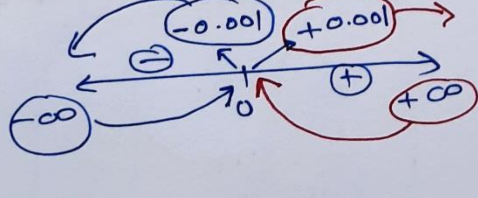
$\frac{1}{0} = \infty$

$\frac{1}{\infty} = 0$

really Output

$x \to$ input

$y, f(x) \to$ output = Range



Range.

$f(x) = \frac{1}{x}$

Domain (input)

$x \in (-\infty, 0) \cup (0, \infty)$

$\frac{1}{x} \in (-\infty, 0) \cup (0, \infty)$

$f(x) \in (-\infty, 0) \cup (0, \infty)$

onto proved

Output

Range $= (-\infty, 0) \cup (0, \infty)$

$= (-\infty, \infty) - \{0\}$

$= R - \{0\}$

$= R_* =$ Codomain

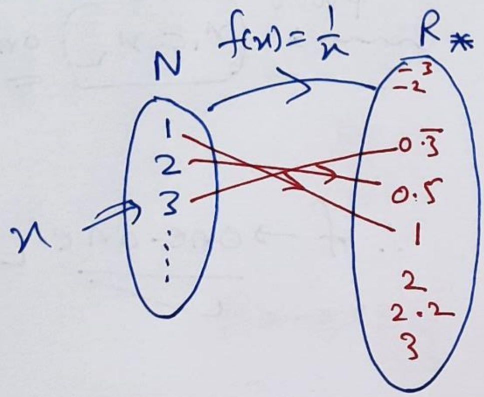
Original function $f : R_* \to R_*$, $f(x) = \frac{1}{x}$ one-one onto

Next Case

$$f : N \to R_* , \quad f(x) = \frac{1}{x} \quad \begin{cases} \to ? \\ \to ? \end{cases}$$

$N = \{1, 2, 3, \ldots\}$

Arrow Diagram.



$f(1) = \frac{1}{1} = 1$

$f(2) = \frac{1}{2} = 0.5$

$f(3) = \frac{1}{3} = 0.\overline{3}$

$f(4) = \frac{1}{4} = 0.25$

one-one ✓

onto → ✗

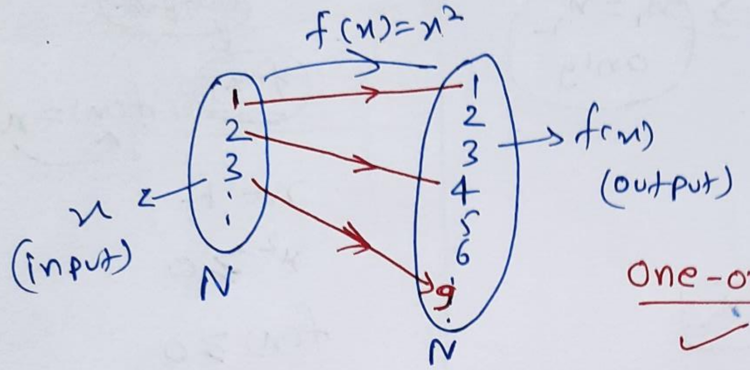
R_* के सारे elements Connected

into

Range ≠ Codomain (R_*)

Q.2 Check injectivity & Surjectivity
(one-one) (onto)
(manyone) (into)

(i) $f: N \to N$ given by $f(x) = x^2$
Codomain



$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

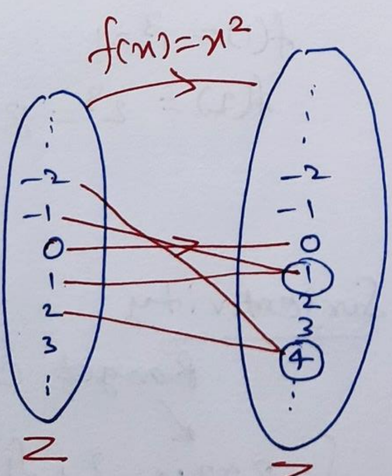
one-one ✓ | onto ✗

Range ≠ Codomain

Range = $\{1, 4, 9, 16, \ldots\} \neq \{1, 2, 3, \ldots\}$ N

one-one & into

(ii) $f: Z \to Z$ given by $f(x) = x^2$
integers



$f(0) = 0^2 = 0$

$f(-1) = (-1)^2 = 1$

$f(-2) = (-2)^2 = 4$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

~~one~~ Injectivity

many-one ✓

Surjectivity

Range ≠ Codomain

$\{0, 1, 4, 9, 16, \ldots\} \neq Z$

x^2

into

(iii) $f: R \to R$ given by $f(x) = x^2$

$(-\infty, \infty)$

<u>One-one</u> $x_1, x_2 \in R$??

$f(x_1) = f(x_2) \to x_1 = x_2$ only

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\boxed{x_1 = x_2}$ या $\boxed{x_1 = -x_2}$

<u>many one</u>

<u>onto</u> Range = Codomain (R)

output

$f(x)$ $f(x) = x^2$

$x \in R$

$x^2 \geq 0$

$f(x) \geq 0$

$f(x) \in [0, \infty)$

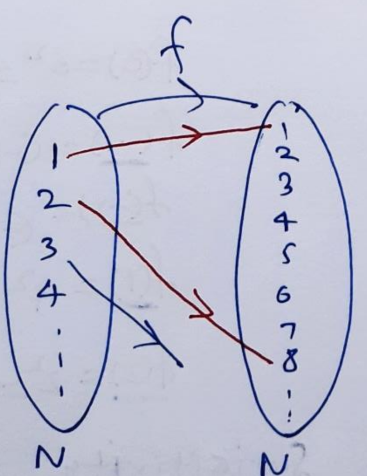
Range $= [0, \infty)$

Codomain $= R \; (-\infty, \infty)$

Range $\neq$ Codomain

Into

(iv) $f: N \to N$ given by $f(x) = x^3$



$f(x) = x^3$

$f(1) = 1^3 = 1$

$f(2) = 2^3 = 8$

<u>One-one</u>

✓

<u>Surjectivity</u>

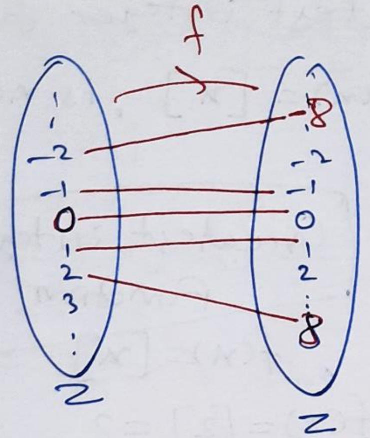
Range $\neq$ Codomain

$\{1, 8, 27, 64 \ldots\} \neq \{1, 2, 3 \ldots\}$ N

into

(V) $f: Z \to Z$ given by $f(x) = x^3$

$\{-2, -1, 0, 1, 2, 3 - -\}$



$f(x) = x^3$

$f(0) = 0$

$f(1) = 1$ | $f(-1) = -1$

$f(2) = 8$ | $f(-2) = -8$

$f(3) = 27$ | $f(-3) = -27$

one-one ✓

onto ✗ into ✓

Range $\neq$ Codomain

$\{ - - 27, 8, 0, -1, -8 \ldots \} \neq (Z)$

Exercise 1.2 (Relations and Functions)

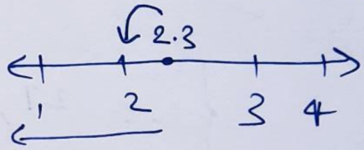
Q.3 Prove that the greatest integer function $f: R \to R$, given by $f(x) = [x]$, is neither one-one nor onto.

Greatest integer function

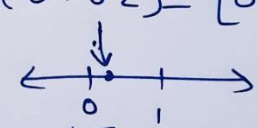
$f(x) = [x]$

$f(2) = [2] = 2$

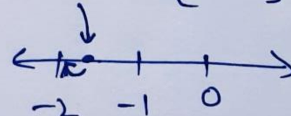
$f(2.3) = [2.3] = 2$



$f(0.02) = [0.02] = 0$



$f(-1.7) = [-1.7] = -2$



one-one $\quad f: R \to R$

$f(x) = [x]$ $\quad$ x input

$x_1 = 8.2$

$f(x_1) = f(8.2) = [8.2] = 8$

$x_2 = 8.5$

$f(x_2) = f(8.5) = [8.5] = 8$

$x_1 \neq x_2$

but $f(x_1) = f(x_2)$

not one-one

onto / into

Range = Codomain

actual output $\quad$ R

$f(x) = [x] =$ Integer $= Z$

Range $= Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

Range $= Z$ $\neq$

Codomain $= R$

into

$\boxed{Q.4}$ $f: R \to R$ $f(x) = |x|$ $\boxed{\text{neither one-one nor onto}}$

one-one.

$f(x_1) = f(x_2) \Rightarrow \boxed{x_1 = x_2}$ only ??

$\Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = \pm x_2$ ⟨ $x_1 = x_2$, $x_1 = -x_2$

$2 = 2$

$|2| = |2|$ ✓

$|-2| = |2|$ ✓

$|-2| = |-2|$

Not one-one

$f(x) = |x|$

$\underline{x_1 = 2} \neq \underline{x_2 = -2}$

$f(x_1) = f(x_2)$

$\downarrow \quad \downarrow$

$2 = 2$

onto

Range = Codomain (R)

↓

output

↓

$f(x) = |x|$

$|x| \geq 0$

$\infty > f(x) \geq 0$

0 ∞

Range $\neq$ Codomain

Range $= [0, \infty)$

Codomain $= R = (-\infty, \infty)$

Into

not onto

$\boxed{Q.5}$ $f: R \to R$ $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ (Signum)

neither oneone nor onto

Not one-one.

$f(2) = 1,\ f(3) = 1$

$f(-2) = -1,\ f(-3) = -1$

$-2, -3 \to 1, 0, -1$

not one-one

many-one

not onto

Range $\neq$ Codomain (R)

↓

Output

↓

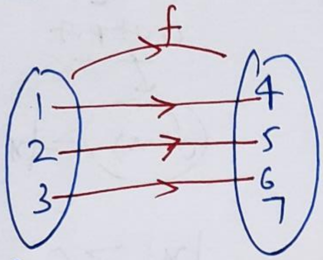
$\{1, 0, -1\}$

not onto

$\boxed{Q.6.}$ $A = \{1, 2, 3\}$ $B = \{4, 5, 6, 7\}$

$f = \{(1,4), (2,5), (3,6)\}$ $f : \underline{A} \to \underline{B}$

$\boxed{\text{Show that } f \text{ is one-one}}$



A B

one – one

$\boxed{Q.7}$ (i) $f : R \to R$ defined by $f(x) = 3 - 4x$

one-one.

Let $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ only

$f(x_1) = f(x_2)$

$\Rightarrow \cancel{3} - 4x_1 = \cancel{3} - 4x_2$

$\Rightarrow \boxed{x_1 = x_2}$ only

$\boxed{\text{one-one}}$

onto $\boxed{\text{Range} = \text{Codomain}}$

Output $(f(x))$

R

$x \in R$

$3 - 4x \in R$

$f(x) \in R$

Range $= R =$ Codomain

onto

Bijective

(ii) $f: R \to R$ defined by $f(x) = 1+x^2$

$$f(x) = 1 + x^2$$

one-one

Let $f(x_1) = f(x_2)$

$\Rightarrow \cancel{1}+x_1^2 = \cancel{1}+x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow \boxed{x_1 = \pm x_2}$

$(x_1 = x_2)$ या $\boxed{x_1 = -x_2}$

not One-one

not ~~Bij~~ Bijective.

onto Range = Codomain

??

Codomain = R → $(-\infty, \infty)$

$x \in R$

$x^2 \geq 0$

$1+x^2 \geq 1+0$

$f(x) \geq 1$

$-\infty$ 1 ∞

$f(x) \in [1, \infty)$

Range

Range $\neq$ Codomain

not onto

Exercise 1.2 (Relations and Functions)

$\boxed{Q.8}$ Let A and B be sets. Show that $f: A \times B \to (B \times A)$ such that $f(a, b) = (b, a)$ is bijective function. one-one & onto

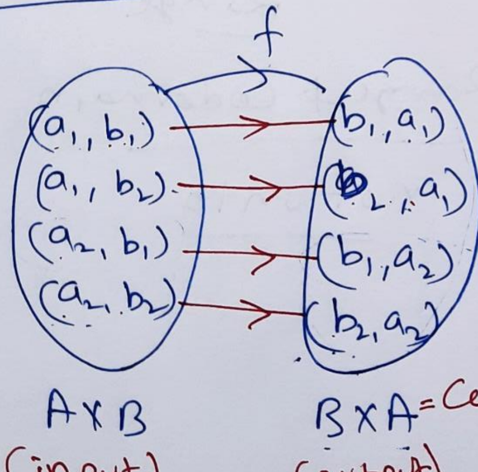
Ans. Let $A = \{a_1, a_2\}$ $B = \{b_1, b_2\}$

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$

Cartesian Product

$B \times A = \{(b_1, a_1), (b_2, a_1), (b_1, a_2), (b_2, a_2)\}$

Arrow Diagram



A×B (input)

B×A = Codomain (output)

$f(a, b) = (b, a)$

$f(x, y) = (y, x)$

$f(1, -3) = (-3, 1)$

$f(a_1, b_1) = (b_1, a_1)$

One-one

✓

✓

onto Range = Codomain

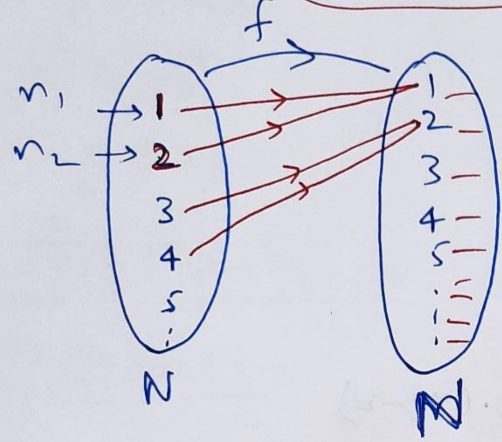
$B \times A = B \times A$

✓

Bijective Fn.

Q.9 $f: N \to N$ be defined by

$$\boxed{f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}}\ ; \ n \in N$$



$f(1) = \frac{1+1}{2} = 1$

$f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2$

$f(4) = \frac{4}{2} = 2$

⋮

One-one → Not

(many-one) ✓

$n_1 = 1, \ n_2 = 2$

$n_1 \neq n_2$ ←

$f(n_1) = f(n_2)$

$f(1) = f(2)$

$\downarrow \quad \downarrow$

$1 \quad 1$

Onto / Into

↓

Range = Codomain

↓ ↓

Output — N ✓

$\{1, 2, 3, \ldots\}$

⇓

N ✓

Onto ✓

not one one ↘ ↙ onto

↓ ↓

Not Bijective

Q.10

$$f : A \rightarrow B \text{ defined by } f(x) = \frac{x-2}{x-3}$$

$A = R - \{3\}$, $B = R - \{1\}$

one-one

let $f(x_1) = f(x_2) \Rightarrow \boxed{x_1 = x_2}$ only ??

let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow (x_1 - 2)\cdot(x_2 - 3) = (x_1 - 3)\cdot(x_2 - 2)$$

$$\Rightarrow \cancel{x_1 x_2} - 3x_1 - 2x_2 + \cancel{6} = \cancel{x_1 x_2} - 2x_1 - 3x_2 + \cancel{6}$$

$$\Rightarrow -3x_1 + 2x_1 = 2x_2 - 3x_2$$

$$\Rightarrow \cancel{-}x_1 = \cancel{-}x_2 \Rightarrow \boxed{x_1 = x_2} \text{ only}$$

$\therefore f(x) \rightarrow$ one-one

onto $\boxed{\text{Range} = \text{Codomain}}$

output $f(x) = y$ $\quad \boxed{B = R - \{1\}}$

$$y = f(x) = \frac{x-2}{x-3} \qquad x - 3 \neq 0, \; x \neq 3$$

$\boxed{\text{represent 'x' in terms of 'y'}}$

$$\Rightarrow xy - 3y = x - 2$$

$$\Rightarrow xy - x = 3y - 2$$

$$x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1} \quad y - 1 \neq 0$$

$y - 1 \neq 0$

$y \neq 1 \quad y \in R - \{1\}$

Range $= R - \{1\}$

Codomain $= R - \{1\}$

Range = codomain

$\boxed{f(x) \rightarrow \text{onto}}$

Q.11 $f: R \to R$ defined by $f(x) = x^4$

one-one / many-one

Let $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ only

let $f(x_1) = f(x_2)$

$\Rightarrow x_1^4 = x_2^4$

$\Rightarrow x_1 = \pm x_2$

$\boxed{x_1 = x_2}$ $\boxed{x_1 = -x_2}$

many-one

Option (D)

onto - into

$\boxed{\text{Range} \neq \text{codomain}}$

R

$x \in R$

$x^4 \geq 0$

$f(x) \geq 0$

$-\infty \quad 0 \quad \infty$

$f(x) \in [0, \infty)$

Range $= [0, \infty)$

Not equal

Codomain $= R = (-\infty, \infty)$

not onto (into)

Q.12 $f: R \to R$ defined by $f(x) = 3x$

one-one / many-one

Let $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ only

let $f(x_1) = f(x_2)$

$3x_1 = 3x_2$

$\Rightarrow \boxed{x_1 = x_2}$

one-one

$\boxed{\text{Option-A}}$

onto - into

$\boxed{\text{Range} = \text{codomain}}$

R

$x \in R$

$3x \in R$

$f(x) \in R$

equal

Range $= R =$ Codomain

onto

Composite Function & Invertible Function

(संयुक्त फलन) (प्रतिलोमीय फलन)

Note: Identity Function (तत्समक फलन) (I)

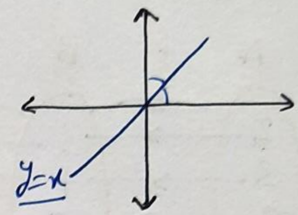
$f(x) = x$

$y = x$

$f : R \to R$ given by $f(x) = x$ $(= I_R)$

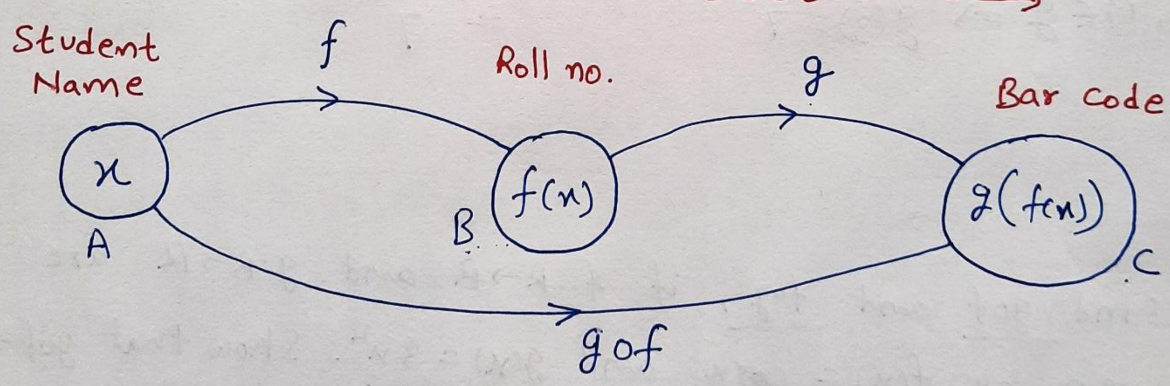
$f : N \to N$ given by $f(x) = x$ $(= I_N)$

$f : A \to A$ given by $f(x) = x$ $(= I_A)$



Composite Function (संयुक्त फलन) ⇒

Copy check.



Definition: Let $f : A \to B$ and $g : B \to C$ be two functions. Then composition of 'f' and 'g' denoted by gof, is defined as the function $gof : A \to C$ given by

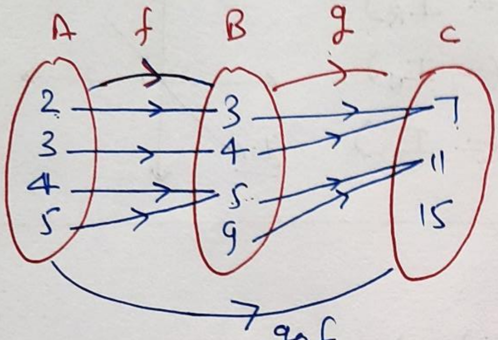
$$gof(x) = g(f(x)), \quad \forall\, x \in A$$

e.g. Let $f : \overset{A}{\{2,3,4,5\}} \to \overset{B}{\{3,4,5,9\}}$ and

$g : \underset{B}{\{3,4,5,9\}} \to \underset{C}{\{7,11,15\}}$ be defined as

$f = \{(2,3), (3,4), (4,5), (5,5)\}$ and

$g = \{(3,7), (4,7), (5,11), (9,11)\}$. Find gof.



$\overset{B\to C}{g}\,o\,\overset{A\to B}{f} = \{(2,7), (3,7), (4,11), (5,11)\}$

$(2,3) \in f \Rightarrow f(2) = 3$

$(3,7) \in g \Rightarrow g(3) = 7$

$gof(2) = g(f(2))$
$= g(3)$
$= 7$

e.g. Find gof and fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

$gof = gof(x)$
$= g(f(x))$
$= g(\cos x)$
$= 3(\cos x)^2$
$= 3\cos^2 x$

$x = 0 \to 3$

$fog = fog(x)$
$= f(g(x))$
$= f(3x^2)$
$= \cos(3x^2)$

$x = 0 \to 1$

$gof \neq fog$

Properties of composite Functions

① $g \circ f \neq f \circ g$ & $h \circ (g \circ f) = (h \circ g) \circ f$

② If f and g are one-one, then $g \circ f$ is $\boxed{\text{one-one}}$.

③ If f and g are onto, then $g \circ f$ is $\boxed{\text{onto}}$.

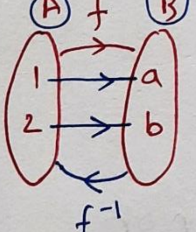
④ If $g \circ f$ is one-one, then 'f' is one-one.

⑤ If $g \circ f$ is onto, then 'g' is onto.

Invertible Functions (प्रतिलोमीय फलन) $[f^{-1} / f^{-1}(x)]$

Eligibility / योग्यता :→ Bijective Function

one-one & onto



one-one: $\boxed{\text{let } f(x_1) = f(x_2) \text{ then } x_1 = x_2 \text{ only}}$

onto: $\boxed{\text{Range} = \text{Codomain}}$

Range: actually output

Codomain: Question $f: A \to B$

Definition: A function $f: X \to Y$ is defined to be invertible, if there exists a function $g: Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

The function 'g' is called the inverse of f.

$$\boxed{g = f^{-1}}$$

$$f \circ f^{-1} = I = f^{-1} \circ f$$

$$\boxed{f \circ f^{-1}(x) = x = f^{-1} \circ f(x)}$$

Process to Find Inverse ($f^{-1}(x)$) of a Function ($f(x)$)

$f(x) =$ expression

- check bijectivity of $f(x)$.

  one-one   onto

- Let $y = f(x) = \boxed{\text{terms}}$ of x
  - write x in terms of y ($x = \boxed{\text{terms}}$ of y)
  - Replace x by $f^{-1}(x)$

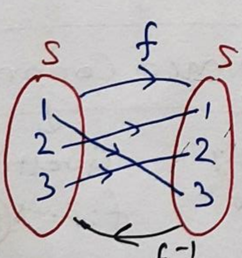
    & y by x.

e.g. Find f^{-1} (if exists).

(i) $S = \{1, 2, 3\}$

$f : S \to S$ given by

$f = \{(1,3), (3,2), (2,1)\}$



$f(1) = 3$

$f(3) = 2$

$f(2) = 1 \Rightarrow f^{-1}(1) = 2$

one-one ✓

onto ✓

Range = Codomain (S)

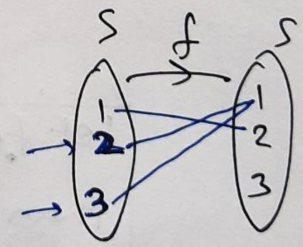
$S = \{1, 2, 3\}$

$f^{-1} = \{(3,1), (2,3), (1,2)\}$

(ii) $S = \{1, 2, 3\}$

$f : S \to S$ given by

$f = \{(1,2), (2,1), (3,1)\}$



one-one ✗ → not bijective → not invertible e.g. $f: R \to R$ defined by $f(x) = 3x - 7$. Find f^{-1} if exists.

Bijectivity

one-one

$f(x_1) = f(x_2)$

$\Rightarrow 3x_1 - \cancel{7} = 3x_2 - \cancel{7}$

$\Rightarrow \boxed{x_1 = x_2}$ only

✓ one-one

onto

Range = Codomain

$f(x)$ R = (R)

$x \in R$

$\Rightarrow 3x - 7 \in R$

$\Rightarrow f(x) \in R$

$\boxed{\text{Range} = R}$

onto ✓

$y = f(x) = 3x - 7$ (let)

$\Rightarrow y = 3x - 7$

$\Rightarrow \frac{y+7}{3} = x$

$\Rightarrow x = \frac{y+7}{3}$

then (after replacement)

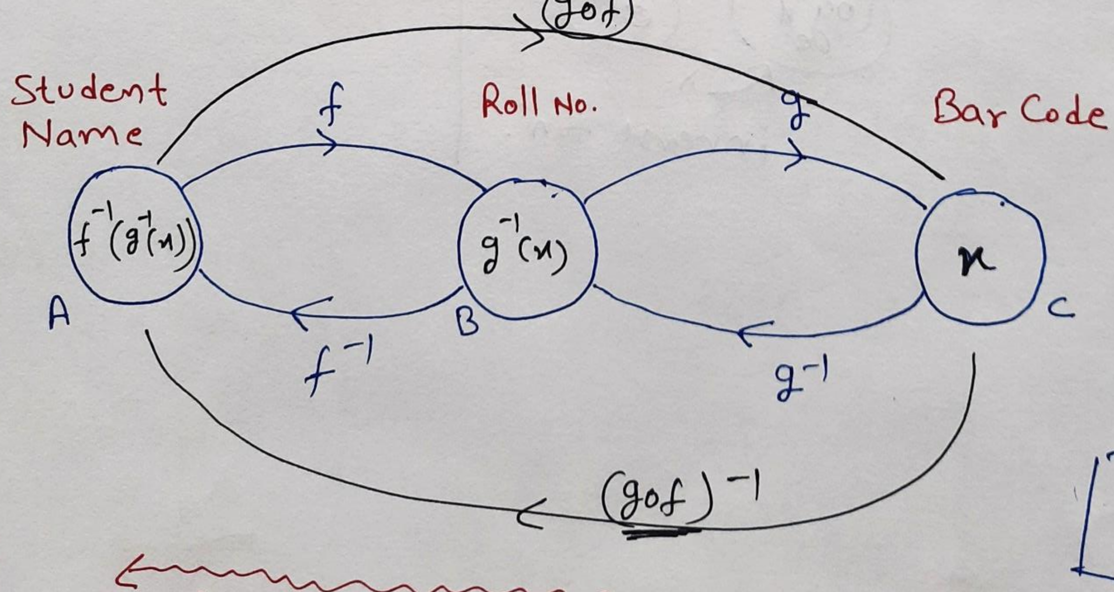
$$\boxed{f^{-1}(x) = \frac{x+7}{3}}$$

Properties of Invertible Functions

① $f \circ f^{-1}(x) = x = f^{-1} \circ f(x)$

② $(f^{-1})^{-1} = f$

☆③ $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ $\quad \boxed{(g \circ f)^{-1}(x) = f^{-1} \circ g^{-1}(x)}$



$\frac{90}{100}$

Marks

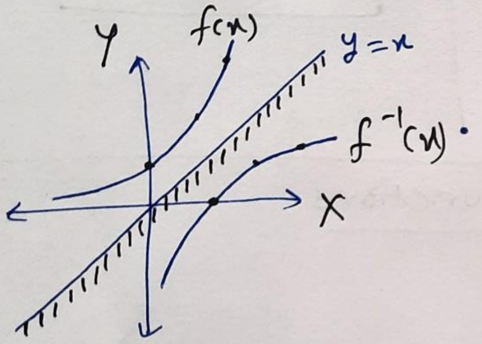
Note: Let $f: X \to Y$ is a bijective function.

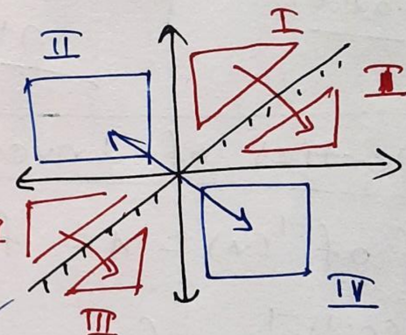
then $f^{-1}: Y \to X$ is a bijective function.

Domain of f = X = Range of f^{-1}

Range of f = Y = Domain of f^{-1}

Graphically, f and f^{-1} are mirror image of each other in mirror line $y = x$.





mirror

$\log_e x$ $\quad$ e^x

inverse funⁿ.

Class 12 Maths (Relⁿ & Funⁿ)

Exercise 1.3 (Composite Funⁿ & Inverse Funⁿ)

Q.1 $f : \overset{A}{\{1,3,4\}} \to \overset{B}{\{1,2,5\}}$, $g : \overset{B}{\{1,2,5\}} \to \overset{C}{\{1,3\}}$ $gof = ?$

$f = \{(1,2), (3,5), (4,1)\}$, $g = \{(1,3), (2,3), (5,1)\}$

$f(1) = 2$, $g(1) = 3$

$f(3) = 5$, $g(2) = 3$

$f(4) = 1$, $g(5) = 1$

$$\boxed{gof = \{(1,3), (3,1), (4,3)\}}$$

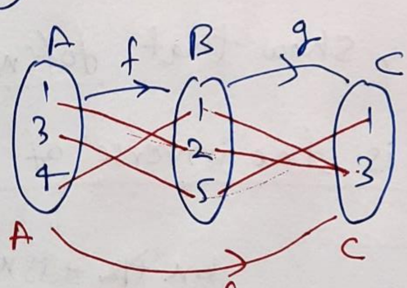
$gof(1) = g(2) = 3$

$gof(3) = g(5) = 1$

$gof(4) = g(1) = 3$

$gof = gof(x)$ $\quad f : A \to B$, $x \mapsto f(x)$

$gof(x)$ — C



$$\boxed{fog = fog(x) = f(g(x))}$$

Composite

Q.2 $f, g, h : R \to R$ Prove that: (i) $(f+g)oh = foh + goh$

(ii) $(f.g)oh = (foh).(goh)$

$(f \pm g)(x) = f(x) \pm g(x)$

$(f.g)(x) = f(x).g(x)$

$(f/g)(x) = \dfrac{f(x)}{g(x)}$

Algebra of Fnⁿ.

(11th)

(i) LHS $= (f+g)oh$

$= (f+g)\,oh\,(x)$

$= (f+g)(h(x))$

$= f(h(x)) + g(h(x))$

$= foh + goh$

$=$ RHS

(ii) LHS $= (f.g)oh$

$= (f.g)(h(x))$

$= f(h(x)).g(h(x))$

$= foh . goh$

$=$ RHS.

(3) Find gof and fog if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

$$||2|| = |2| = 2$$
$$||-2|| = |2| = 2$$

$$gof = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$fog = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$$

(ii) $f(x) = 8x^3$, $g(x) = x^{1/3}$

$$gof = gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{1/3} = 8^{\frac{1}{3}} \cdot (x^3)^{\frac{1}{3}}$$
$$= 2x$$

$$fog = f(g(x)) = f(x^{\frac{1}{3}}) = 8(x^{1/3})^3 = 8x$$

(4) If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $fof(x) = x$, for all $x \neq \frac{2}{3}$. what is the inverse of 'f'?

To Prove: $\boxed{fof(x) = x}$

$$LHS = fof(x)$$
$$= f(f(x))$$
$$= f\left(\frac{4x+3}{6x-4}\right)$$

$$fof(x) = \frac{4\left[\frac{4x+3}{6x-4}\right] + 3}{6\left[\frac{4x+3}{6x-4}\right] - 4}$$

$$= \frac{\frac{16x + \cancel{12} + 18x - \cancel{12}}{\cancel{6x-4}}}{\frac{\cancel{24x} + 18 - \cancel{24x} + 16}{\cancel{6x-4}}}$$

$$= \frac{34x}{34} = x = RHS$$

$f: X \to Y$, $f^{-1}: Y \to X$

$fof^{-1}(x) = x = f^{-1}of(x)$

$fog(x) = x = gof(x)$

Let

$y = f(x) = \dfrac{4x+3}{6x-4}$

① Let $y = f(x) = \ldots$

② x in terms of y

$x = \ldots (y)$

③ $x \to f^{-1}(x)$

$y \to x$

$\Rightarrow y = \dfrac{4x+3}{6x-4}$

$\Rightarrow 6xy - 4y = 4x + 3$

$\Rightarrow 6xy - 4x = 4y + 3$

$\Rightarrow x(6y-4) = 4y+3$

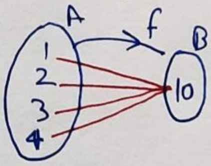
$\Rightarrow x = \dfrac{4y+3}{6y-4} \Rightarrow \boxed{f^{-1}(x) = \dfrac{4x+3}{6x-4}} = f(x)$ ✓

Q.5 Invertible < Yes. / No. (with reason)

Bijective: one-one, onto

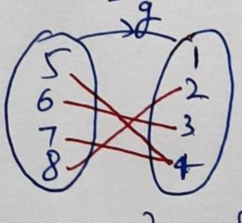
(i) $f: \{1,2,3,4\} \to \{10\}$, $f = \{(1,10),(2,10),(3,10),(4,10)\}$

A = $\{1,2,3,4\}$, B = $\{10\}$



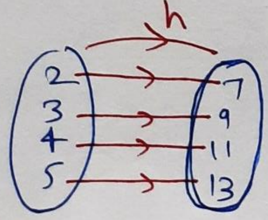
many-one Not invertible

(ii) $g: \{5,6,7,8\} \to \{1,2,3,4\}$, $g = \{(5,4),(6,3),(7,4),(8,2)\}$



many-one & into Not invertible

(iii) $h: \{2,3,4,5\} \to \{7,9,11,13\}$, $h = \{(2,7),(3,9),(4,11),(5,13)\}$



one-one, onto Yes

Range = codomain

(12^{th} maths) (Relations & Functions)

Exercise 1.3 $\boxed{\text{Composite Funⁿ. \& Inverse Funⁿ.}}$

Q.6 Show that $f:[-1,1]\to R$, given by $f(x)=\frac{x}{x+2}$ is one-one. Find the inverse of the function $f:[-1,1]\to$ Range of f.

Ans. $f(x)=\frac{x}{x+2}$

one-one: let $f(x_1)=f(x_2) \Rightarrow \boxed{x_1=x_2}$ only

$$\Rightarrow \frac{x_1}{x_1+2}=\frac{x_2}{x_2+2}$$

$$\Rightarrow \cancel{x_1x_2}+2x_1=\cancel{x_1x_2}+2x_2$$

$$\Rightarrow \boxed{x_1=x_2} \text{ only} \rightarrow f(x)\rightarrow \text{one-one.}$$

Bijective → one-one ✓, onto

onto: Range = Codomain ✓ → Range of f

$f:A\to B$ (B: Codomain)

Inverse of $f(x)=\frac{x}{x+2}$

① $y=f(x)$ (माना)
② x in terms of 'y'
③ Replace $x\to f^{-1}(x)$, $y\to x$

Let $y=f(x)=\frac{x}{x+2}$

$$\Rightarrow y=\frac{x}{x+2}$$

$$\Rightarrow xy+2y=x$$

$$\Rightarrow xy-x=-2y$$

$$\Rightarrow x(y-1)=-2y$$

$$\Rightarrow x=\frac{-2y}{y-1}$$

$$\Rightarrow f^{-1}(x)=\frac{-2x}{x-1}=\frac{2x}{1-x}$$

Domain

Q.7 Consider $f: R \to R$ given by $f(x) = 4x+3$. Show that f is invertible. Find the inverse of f.

Bijective

one-one

onto $\boxed{\text{Range} = \text{codomain}}$

? (Actual output) (R)

let $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + \cancel{3} = 4x_2 + \cancel{3}$

$\Rightarrow \boxed{x_1 = x_2}$ only

one-one ✓

$x \to$ input $x \in R$
$f(x) \to$ output

$x \in R$

$\Rightarrow 4x+3 \in R$

$\Rightarrow f(x) \in R$ Range $= R =$ Codomain

onto ✓

$\therefore f$ is invertible,

$f(x) = 4x+3 = y$ (let)

$\Rightarrow 4x = y-3$

$\Rightarrow x = \frac{y-3}{4}$

Replace $x \to y \to f^{-1}(x)$

$y \to x$

$\Rightarrow \boxed{f^{-1}(x) = \frac{x-3}{4}}$ ✓

$\boxed{gof(x) = x = fog(x) \quad g \to f^{-1}}$

$\boxed{Q.8}$ Consider $f: R_+ \to [4,\infty)$ given by $f(x) = x^2 + 4$.
Show that f is invertible with $f^{-1}(y) = \sqrt{y-4}$,
where R^+ is the set of all non-negative real no.

(x) Domain $= R_+ = [0,\infty) \to x_1, \to x_2$

Codomain $= [4,\infty)$

$\overset{-}{\times}$ 0 $\overset{+}{\checkmark}$ ∞

$R^+ = [0,\infty)$

Invertible $\to$ Bijective

one-one

$\boxed{\text{Let } f(x_1) = f(x_2) \Rightarrow \text{only } x_1 = x_2}$

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + \cancel{4} = x_2^2 + \cancel{4}$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\boxed{x_1 = x_2}$ ✓

$\boxed{x_1 = -x_2}$ ×

$x_2 \to R^+$

$x_1 \to \ominus$

×

One-one

onto

$\boxed{\text{Range} = \text{Codomain}}$

? $\quad [4,\infty)$

$f(x) \to x^2 + 4$

$\because x \in [0,\infty)$

$\Rightarrow x^2 \in [0,\infty)$

$\Rightarrow x^2 + 4 \in [4,\infty)$

$f(x) \in [4,\infty)$

Range $= [4,\infty) =$ Codomain

onto ✓

Let $y = f(x) = x^2 + 4$

$\Rightarrow y = x^2 + 4$

$\Rightarrow y - 4 = x^2$

$\Rightarrow x = \pm\sqrt{y-4}$

$\because x \in R^+ \Rightarrow x = \sqrt{y-4}$

Replacement.

$f^{-1}(x) = \sqrt{x-4}$

$f^{-1}(100) = \sqrt{100-4}$

$f^{-1}(y) = \sqrt{y-4}$ ✓

Q.9 $f : R_+ \to [-5, \infty)$ $f(x) = 9x^2 + 6x - 5$

Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6} - 1}{3}\right)$.

(x) Domain $= R_+ = [0, \infty)$

Codomain $= [-5, \infty)$

Invertible

↓

Bijective.

One-one

$\boxed{\text{let } f(x_1) = f(x_2) \Rightarrow \boxed{x_1 = x_2} \text{ only}}$

$\Rightarrow 9x_1^2 + 6x_1 - \cancel{5} = 9x_2^2 + 6x_2 - \cancel{5}$

$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow 3(x_1 - x_2)[3(x_1 + x_2) + 2] = 0$

0 | 0

$\boxed{x_1 = x_2}$ only ✓ Accept

$3(x_1 + x_2) + 2 = 0$ ×

$\Rightarrow 3x_1 + 3x_2 + 2 = 0$

$\Rightarrow 3x_1 = -3x_2 - 2$

$\Rightarrow x_1 = \frac{-3x_2 - 2}{3}$

$x_2 \to (+)$

$x_1 \to (-)$

$\boxed{\begin{array}{l} x \in R_+ \\ x_1 \in R_+ \\ x_2 \in R_+ \end{array}}$

Given.

Reject

One-one.

Onto

$\boxed{\text{Range} = \text{Codomain}}$

? $[-5, \infty)$

$f(x) = 9x^2 + 6x - 5$ → Perfect Square.

$f(x) = 9x^2 + 6x - 5$

$= \underbrace{(3x)^2 + 2.(3x).1 + 1^2}_{} - 1 - 5$

$f(x) = (3x+1)^2 - 6$

$x \in [0, \infty)$

$\Rightarrow 3x \in [0, \infty)$

$\Rightarrow 3x + 1 \in [1, \infty)$

$\Rightarrow (3x+1)^2 \in [1, \infty)$

$\Rightarrow (3x+1)^2 - 6 \in [-5, \infty)$

$f(x) \in [-5, \infty)$

Range $= [-5, \infty) =$ Codomain

Onto

$$\underbrace{y = f(x) = 9x^2 + 6x - 5 = (3x+1)^2 - 6}$$

① $y = f(x)$

② x in terms of y

③ $\cancel{x \to f^{-1}(x)}$

$$y = (3x+1)^2 - 6$$

$$\Rightarrow y + 6 = (3x+1)^2$$

$$\Rightarrow \pm\sqrt{y+6} = 3x+1$$

$$\Rightarrow \pm\sqrt{y+6} - 1 = 3x$$

$$\Rightarrow x = \frac{\pm\sqrt{y+6} - 1}{3}$$

$$\boxed{\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \end{aligned}}$$

$\because x \in R^+$

$$x = \frac{\sqrt{y+6} - 1}{3} \Rightarrow \boxed{f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}}$$

Exercise 1.3 {Composite, Inverse Functions}

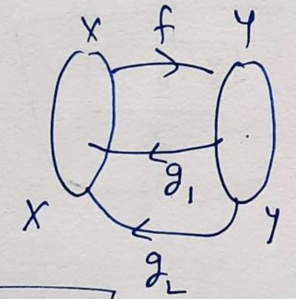
Q.10 Let $f: X \to Y$ be an invertible function. Show that f has unique inverse.

Proof: (Proof by Contradiction)

Let f does not have a unique inverse

Let inverse of function f are g_1 & g_2.

$g_1 \neq g_2$.



$f \to$ invertible

Bijective

one-one

onto

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\boxed{g_1 : Y \to X}$ also $\boxed{g_2 : Y \to X}$

$f o g_1(x) = I_y = f o g_2(x)$ ← By Definition of Inverse.

$\Rightarrow f o g_1(x) = f o g_2(x)$

$\Rightarrow f(g_1(x)) = f(g_2(x))$ $(\because f \to \text{one-one})$

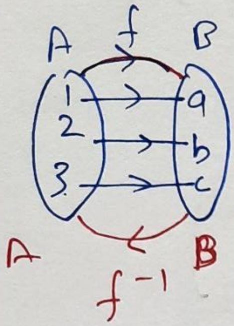
$\Rightarrow g_1(x) = g_2(x)$

$\Rightarrow g_1 = g_2$ This contradicts our assumption.

$\therefore f$ has a unique inverse.

Q.11 $f : \overset{A}{\{1,2,3\}} \to \overset{B}{\{a,b,c\}}$ $f(1) = a$, $f(2) = b$, $f(3) = c$.

Find f^{-1}. show that $(f^{-1})^{-1} = f$



$f = \{(1,a), (2,b), (3,c)\}$

$\therefore f^{-1} = \{(a,1), (b,2), (c,3)\} = g$

$(f^{-1})^{-1} = \{(1,a), (2,b), (3,c)\} = g^{-1}$

$\boxed{(f^{-1})^{-1} = f}$

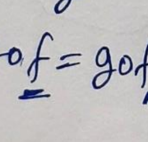
Q.12 Let $f : X \to Y$ be an invertible function. Show that inverse of f^{-1} is f i.e. $(f^{-1})^{-1} = f$.

Ans. $(f : X \to Y)$

Inverse of $f = f^{-1} = g$ (let) $\quad \begin{pmatrix} f^{-1} : Y \to X \\ g : Y \to X \end{pmatrix}$

By definition,

$g \circ f = I = f \circ g$ $\quad$ $f(x) = x$ $\quad$ $I \to$ Identity Fn. $f(x) = x$



$g \circ f = I_X$ $\qquad$ $f \circ g = I_Y$

$g \circ f = I_x$ $\qquad$ $f \circ g = x$

$\Rightarrow g \circ f(x) = x$

$g \circ f = g \circ f(x)$

$\Rightarrow f(x) = g^{-1}(x)$

$f : X \to X$

$\Rightarrow f(x) = (f^{-1})^{-1}(x)$ $\quad \Big| \quad \Rightarrow f = (f^{-1})^{-1}$

Q.13 If $f: R \to R$ be given by $f(x) = (3-x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $x^{1/3}$ (B) x^3 (C) x (D) $3-x^3$

Ans. $f(x) = (3-x^3)^{1/3}$

$fof(x) = f(f(x)) = f\left((3-x^3)^{1/3}\right)$

$= \left(3 - \left[(3-x^3)^{1/3}\right]^3\right)^{1/3}$

$= \left[3 - (3-x^3)\right]^{1/3} = (\cancel{3} - \cancel{3} + x^3)^{1/3}$

$fof(x) = x$

Q.14 $f(x) = \dfrac{4x}{3x+4}$

(A) $g(y) = \dfrac{3y}{3-4y}$ (B) $g(y) = \dfrac{4y}{4-3y}$

(C) $g(y) = \dfrac{4y}{3-4y}$ (D) $g(y) = \dfrac{3y}{4-3y}$

$f^{-1} = g$

① $y = f(x)$

② $x \to y$ Terms

$\Rightarrow y = f(x) = \dfrac{4x}{3x+4}$

$\Rightarrow y = \dfrac{4x}{3x+4}$

$\Rightarrow 3xy + 4y = 4x$

$\Rightarrow 4y = 4x - 3xy$

$\Rightarrow 4y = x.(4-3y)$

$\Rightarrow x = \dfrac{4y}{4-3y}$

$\Rightarrow f^{-1}(y) = \dfrac{4y}{4-3y}$

$\Rightarrow \boxed{g(y) = \dfrac{4y}{4-3y}}$

$y = f(x)$

$f^{-1}(y) = x$

Class 12 Maths [Relations & Functions]

Binary Operations [द्विआधारी संक्रियाएं]

For State Boards

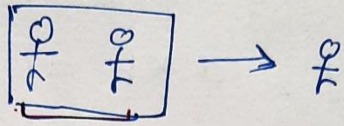
Binary → 2 दो

$\oplus \quad \ominus \quad \otimes \quad \oplus\!\!\!\!\div$

$\underbrace{\underbrace{100 + 35}_{} + 98}_{}$

~~+++++~~

$\underline{3 \quad 5} \quad 7$



Set = Humans

Definition of Binary operations (*) on A → from A to A

A binary operation '*' on a set A is a function $* : A \times A \to A$.

$* : \underline{A \times A} \to \underline{A}$

We denote $*(a,b)$ by $a * b$.

$*(a,b)$
↓
$f(a,b)$

$*: A \times A \to A$

(A) a → $\boxed{a * b}$ → c
(A) b

(A)

2 input

(1 output)

e.g. $+(a,b) \quad a, b \in N \;\; (2, 3) \;\; \to \;\; +(a,b) = +(2,3) = 2+3 = 5$

N N N

e.g. Check, which of the following $*$ is binary?

(i) On Z, $a * b = a - b$

(ii) on N, $a * b = a - b$

(iii) On Q, $a * b = a + (a+b)^2$

(iv) On $R - \{-1\}$, $a * b = \frac{a}{b+1}$

(i) Z = integers $= \{\ldots -2, -1, 0, 1, 2 \ldots\}$

$a * b = a - b$ $a \in Z$, $b \in Z$ $\therefore a - b \in Z$ ✓

$* \longrightarrow$ Binary ✓

(ii) $N = \{1, 2, 3, \ldots\}$ $a \in N$, $b \in N$, $a - b \notin N$

$a * b = a - b$ $a = 1$, $b = 5$

$1 * 5 = 1 - 5 = -4 \notin N$ Not Binary

(iii) Q = Rational no. = परिमेय सं. $= \frac{p}{q}$ $q \neq 0$

$a * b = a + (a+b)^2$, $a \in Q$, $b \in Q$, $a + (a+b)^2 \in Q$

Binary ✓

IV $R - \{-1\}$

2 inputs 1 output

$a * b = \frac{a}{b+1}$

$a \in R - \{-1\}$ ✓

$b \in R - \{-1\}$ ✓

$\boxed{a = -\frac{1}{2}, b = -\frac{1}{2}}$

$\left(-\frac{1}{2}\right) * \left(-\frac{1}{2}\right) = \frac{-\frac{1}{2}}{-\frac{1}{2}+1} = \frac{\left(-\cancel{\frac{1}{2}}\right)}{\left(\cancel{\frac{1}{2}}\right)} = -1$

$\frac{a}{b+1} \notin R - \{-1\}$

Not Binary

Commutativity (क्रम विनिमेयता)

if $a * b = b * a$

Associativity (साहचर्यता) if $(a * b) * c = a * (b * c)$

Identity (तत्समक) $= e$ (identity of $*$)

$a * e = a = e * a$

Unique

Inverse (प्रतिलोम) $= b$ (माना) (Let)

$a * b = e = b * a$

(inverse of a)

e.g. Consider a binary operation '$*$' on the set $\{1, 2, 3\}$ given by following operation table

$*$	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

$a * b$

(i) Compute $(2 * 3) * 1$

$= (3) * 1 = 3$

(ii) Compute $2 * (3 * 1)$

$= 2 * (3)$

$= 3$

(iii) Is $*$ commutative?

$a * b = b * a$

$2 * 3 = 3 * 2$

$1 * 3 = 3 * 1$

$1 * 2 = 2 * 1$

e.g. Let $*$ be a binary operation on the set $\{1,2,3,4,5\}$ defined by

$\boxed{a * b = \text{HCF of } a \,\&\, b}$.

(i) Is $*$ a binary operations?

(ii) Is $*$ commutative?

(iii) find identity element for $*$?

Ans: $a * b = HCF(a,b)$ $\quad A = \{1,2,3,4,5\}$

A A A

$1 * 2 = HCF(1,2) = 1 \in A$

$2 * 4 = HCF(2,4) = 2 \in A$

$3 * 5 = HCF(3,5) = 1 \in A$

(i) Yes. (Binary ✓)

(ii) $a * b = \underline{\text{HCF of } a \,\&\, b} = \underline{\text{HCF of } b \,\&\, a}$

$= b * a$

Yes Commutative

(iii) Identity Element $= e$

$a * e = a = e * a$ $\quad a \in A$

$a \in \{1,2,3,4,5\}$

$a = 5$

$5 * e = 5$

$\Rightarrow$ HCF of $5 \,\&\, e = 5$ $\quad$ HCF 5 & 5

$e = 5$

$e = 5 \quad e = 4$

Not unique

$a = 4$

$4 * e = 4$

$\Rightarrow$ HCF of $4 \,\&\, e = 4$

$\Rightarrow$ $e = 4$

$\boxed{\text{Identity element does not exist}}$