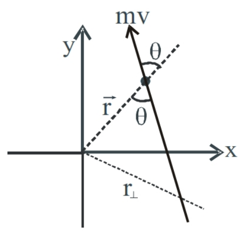
Rotational Motion

Angular Momentum



Angular momentum of a rotating body about a point is given by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

Work done in rotational motion in terms of torque

Work done in rotational motion , $\sum\left(\vec{r}_i \times \vec{F}_i\right)\Delta\theta$

where $\sum \vec{r}_i \times \vec{F}_i$ is the algebraic sum of moment of force and $\Delta\theta$ is the angle through which a body is rotated.

$\therefore \quad \Delta W = \text{Total torque} \times \text{angular displacement} \Rightarrow d\omega = \tau d\theta$

$\therefore \quad \omega = \int d\omega = \int \tau . d\theta$, τ is the instantaneous torque.

Equations of rotational motion as compared to linear motion

(i) $\omega^2 - \omega_0^2 = 2\alpha\theta$ comparable to $v^2 – u^2 = 2as$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ comparable to $s = ut + \frac{1}{2}at^2$

(iii) $\omega = \omega_0 + \alpha t$ comparable to $v = u + at$

where ω_0 is initial angular velocity, ω is final angular velocity, α is angular acceleration and θ is angular displacement.

Moment of Inertia is a measure of rotational inertia of a body

Moment of inertia of a system about an axis of rotation is the sum of the product of massess of its particles and squares of their normal distances from the axis i.e.,

$$I = \sum_{i=1}^{i=n} m_i r_i^2 \quad \text{and} \quad I = \int r^2 dm \text{ for rigid body.}$$

Higher the value of moment of inertia, more difficult is the change of state of rotation. Moment of inertia is neither a scalar nor a vector but it is a tensor.

S.I. unit of moment of inertia is kg m^2. Dimensional formula of moment of inertia is $[ML^2T^0]$.

Factors on which moment of inertia depend

Moment of inertia depend upon

(i) distribution of mass of the body about the axis of the rotation

(ii) shape and size of the body

(iii) position and orientation of the axis of rotation

Kinetic enery of rotation

The energy of a body due to its rotational motion is called kinetic rotational energy.

$$K.E._{rot} = \frac{1}{2}I\omega^2 \qquad I = \text{M.I. about axis of rotation.}; \ \omega = \text{angular speed about axis.}$$

Moment of inertia in terms of radius of gyration

Radius of gyration (K) : It is the perpendicular distance of which a point mass equal to mass of system for which moment of inertia of the point mass as same as moment of inertia of the system about the axis. The distance is called radius of gyration of the system.

$$I = MK^2 \quad \text{and} \quad K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}}$$

Moment of inertia is the product of the mass of a body and the square of the radius of gyration.

Relation between moment of inertia and torque

As $\quad d\omega = \tau d\theta$

$$\frac{d\omega}{dt} = \frac{d\tau d\theta}{dt} = \tau\omega = \frac{d\left(\frac{1}{2}I\omega^2\right)}{dt} = (I\alpha)\omega$$

$$\Rightarrow \tau = I\alpha$$

∴ Moment of inertia of a body an axis is numerically equal to the torque acting on it when the body is rotating with a unit angular acceleration.

Vectorially, $\quad \vec{\tau} = I\vec{\alpha}$

The above relation is called Law of rotation or basic equation of rotation

Relation between moment of inertia and angular momentum

Angular momentum of a particle is given by the product of linear momentum and perpendicular distance of the paticle from the axis of rotation. $L = I\omega$

$$\vec{\tau} = \frac{d\vec{t}}{dt} \text{ and } \vec{\tau} = I\vec{r} = \frac{d}{dt}(I\vec{\omega})$$

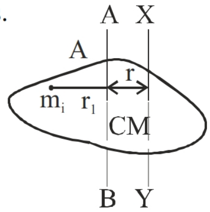
$$\Rightarrow \vec{L} = I\vec{\omega}$$

∴ Moment of inertia of a body about an axis of rotation is equal to the angular momentum of the body about the some axis rotating with unit angular velocity.

Theorem of parallel axes

This theorem states that moment of inertia of a rigid body about an axis (I_{11}) parallel to an axis passing through centre of mass (I_{CM}) of the body is equal to moment of inertia about this axis plus the product of total mass M of the body and square of perpendicular distance (r) between these parallel axes.

i.e., $\quad I_{||} = I_{CM} + Mr^2$



Theorem of perependicular axes

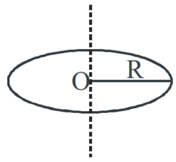
The sum of moments of inertia along two mutually perpendicular axes in a plane is equal to the moment of inertia along the axes perpendicular to this plane

i.e. $\quad I_z = I_x + I_y$

This theorem is only applicable for plane lamina (Mass is distributed along two dimension)

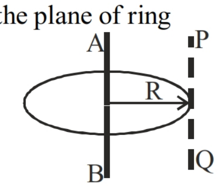
Moment of inertia of a thin uniform ring

(a) Moment of inertia of a thin uniform ring about an axis passing through centre of mass and perpendicular to the plane of the ring is I= MR^2.



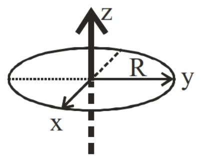
(b) Moment of inertia of a uniform ring about a tangent normal to the plane of ring

$$I_{PQ} = 2MR^2$$



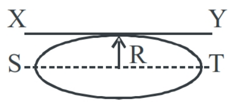
(c) Moment of inertia of a uniform ring abou any diameter of the ring

$$I_x = \frac{MR^2}{2}$$



(d) Moment of inertia of a uniform ring about a tangent in the plane of the ring.

$$I_{XY} = \frac{3}{2}MR^2$$



Moment of Inertia of a uniform disc

Moment of inertia of a disc about line perpendicular at centre of disc $= \frac{MR^2}{2}$

Moment of inertia of about any diameter of disc $= \frac{MR^2}{4}$

Moment of inertia of a uniform cylinder

Moment of inertia of a uniform solid cylinder about its axis $= \frac{MR^2}{2}$

Moment of inertia of a hollow cylinder about its axis = MR^2

Moment of Inertia of a sphere about any diameter

Consider a sphere of mass M and radius R. Moment of inertia of the whole sphere about any diameter

$$I = \frac{2}{5}MR^2$$

Moment of inertia of a hollow sphere about any diameter = $\frac{2}{3}MR^2$

Moment of inertia of a thin rod about an axis passing

Moment of inertia of a thin rod of mass M and length l about a line perpendicular at one of its end $= \frac{ML^2}{3}$

Uniform rectangular lamina of length a and breadth b about perpendicular axis to the plane of lamina and through its centre.

$$= M\left(\frac{a^2+b^2}{12}\right)$$

Acceleration of the centre of mass

Acceleration, $\vec{a} = \frac{d^2\vec{r}_{CM}}{dt^2}$, where $\vec{r}_{CM}$ is position vector of centre of mass whose magnitude is given by

$$\vec{r}_{CM} = \left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}\right)$$

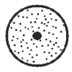
Expression for external force in terms of acceleration of centre of mass

External force, $\vec{F} = \frac{Md^2\vec{r}_{CM}}{dt^2}$, where $\vec{r}_{CM}$ is position vector of centre of mass.

Position of centre of mass in the case of following rigid bodies

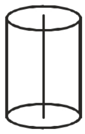
A sphere, a ring, a cylinder, a cone and a cube

A sphere – centre of the sphere

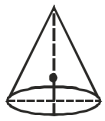


A ring – centre of the ring

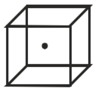
A cylinder – Middle point onthe axis of the cylinder



A solid cone – At a point distant $\frac{1}{4}$th of height of the cone from the base of the axis of the cone



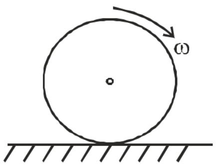
A cube – Point of intersection of diagonals



ROLLING MOTION ON HORIZONTAL ROUGH SURFACE

Pure rolling motion can be considered as a pure rotation about instantaneous axis of rotation.

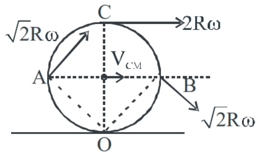
Linear velocity at top point = $2R\omega$



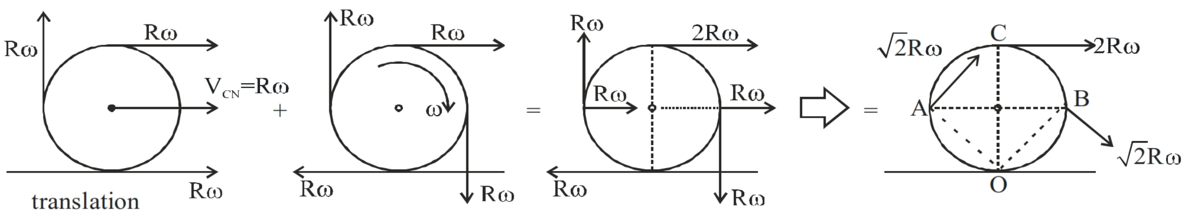
$$V_A = V_B = \sqrt{2}R\omega$$

$V_0 = 0$,

Direction of velocity at a point is perpendicular to the chord joining point of contact to the point.



Rolling motion is considered as the combnation of translation of centre of mass and rotation about centre of mass.



$V_C = 2R\omega$ $\qquad$ $V_B = V_A = \sqrt{2}R\omega$

$V_0 = 0$, hence point of contact has rolling motion on the inclined plane.

Cylinder rolling down without slipping down a rough inclinde plane.

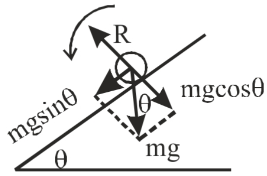
Acceleration, $$a = \frac{mg\sin\theta}{m + I/r^2} = \frac{g\sin\theta}{1+\left(\frac{k}{r}\right)^2}$$

where $mK^2 = I$, K = radius gyration, r = radius of body.

Hence for $\frac{k}{r}$ lesser, then acceleration 'a' down the plane is more so time required to come down is also more.

For pure rolling down the plane, $F_S \le u_S N \Rightarrow I_{CM}\frac{a}{R^2} \le u_S N$

$$\Rightarrow \frac{I_{CM}}{R^2}\frac{g\sin\theta}{1+\left(\frac{k}{r}\right)^2} \le \mu_S mg\cos\theta$$



$$\Rightarrow \quad \mu_S \ge \frac{\tan\theta}{1+\left(\frac{r}{k}\right)^2}$$

For pure rolling $\mu_S \ge \frac{\tan\theta}{1+\left(\frac{r}{k}\right)^2}$ condition for pure rolling down the plane. For disc, $\left(\frac{r}{k}\right)^2 = 2 \therefore \mu_S > \frac{1}{3}\tan\theta$

When body either rolls up or down the inclined plane the friction (static) acts on the body at the point of contact always up the plane. However, no power is dissipated by friction hence mechanical energy is conserved.

CONSERVATION OF ANGULAR MOMENTUM

As $\vec{\tau}_E \text{total} = \frac{d\vec{L}}{dt}$ i.e. rate of change of angular momentum is equal to the total internal torque on the system, where direction of torque is in the direction of change of angular momentum. If there is no torque about a point or line then there is angular momentum is conserved about that point.

In case when body rolls or rolls with slipping then about point of contact there is no torque hence total angular momentum is conserved.